Diffusion Schemes and Variance Positivity

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1 Milstein

1.1 Scheme

Let's consider the following stochastic process

$$dX_t = b(X_t)dt + \sigma(X_t)dW_t$$

between t_k and t_{k+1} we have:

$$X_{t_{k+1}} = X_{t_k} + \int_{t_k}^{t_{k+1}} b(X_t)dt + \int_{t_k}^{t_{k+1}} \sigma(X_t)dW_t \tag{1}$$

applying Itô on σ between t_k et t we get:

$$\begin{aligned}\sigma(X_t) &= \sigma(X_{t_k}) + \int_{t_k}^{t} \sigma\prime(X_s)dX_s + \frac{1}{2}\int_{t_k}^{t} \sigma\prime\prime(X_s)d < X, X >_s \\ &= \sigma(X_{t_k}) + \int_{t_k}^{t} \sigma\prime(X_s)(b(X_s)dt + \sigma(X_s)dW_s) + \frac{1}{2}\int_{t_k}^{t} \sigma\prime\prime(X_s)\sigma^2(X_s)ds\end{aligned}$$

We only keep terms in order $\frac{1}{2}$ with respect to Δt. We get

$$\begin{aligned}\sigma(X_t) &\simeq \sigma(X_{t_k}) + \int_{t_k}^{t} \sigma\prime(X_s)\sigma(X_s)dW_s \\ &\simeq \sigma(X_{t_k}) + \sigma\prime(X_{t_k})\sigma(X_{t_k})(W_t - W_{t_k})\end{aligned}$$

Replacing in **(1)** we get

$$X_{t_{k+1}} \simeq X_{t_k} + \int_{t_k}^{t_{k+1}} b(X_t)dt + \sigma(X_{t_k})(W_{t_{k+1}} - W_{t_k}) + \sigma\prime(X_{t_k})\sigma(X_{t_k})\int_{t_k}^{t_{k+1}} (W_t - W_{t_k})dW_t \tag{2}$$

We recall that

$$d(W_t^2) = 2W_t dW_t + dt$$

Then we easyly compute

$$\int_{t_k}^{t_{k+1}} (W_t - W_{t_k})dW_t = \frac{1}{2}((W_{t_{k+1}} - W_{t_k})^2 - \Delta t)$$

We replace in **(2)**, and we get

$$X_{t_{k+1}} \simeq X_{t_k} + \int_{t_k}^{t_{k+1}} b(X_t)dt + \sigma(X_{t_k})(W_{t_{k+1}} - W_{t_k}) + \frac{1}{2}\sigma\prime(X_{t_k})\sigma(X_{t_k})((W_{t_{k+1}} - W_{t_k})^2 - \Delta t)$$

Depending on the proxy we choose for $\int_{t_k}^{t_{k+1}} b(X_t)dt$, we get the explicit Milstein, or the implicit Milstein scheme.

For

$$\int_{t_k}^{t_{k+1}} b(X_t)dt \simeq b(X_{t_k})\Delta t$$

we get the **explicit Milstein Scheme**:

$$X_{t_{k+1}} \simeq X_{t_k} + b(X_{t_k})\Delta t + \sigma(X_{t_k})(W_{t_{k+1}} - W_{t_k}) + \frac{1}{2}\sigma\prime(X_{t_k})\sigma(X_{t_k})((W_{t_{k+1}} - W_{t_k})^2 - \Delta t)$$

And for

$$\int_{t_k}^{t_{k+1}} b(X_t)dt \simeq b(X_{t_{k+1}})\Delta t$$

we get the **implicit Milstein Scheme**:

$$X_{t_{k+1}} \simeq X_{t_k} + b(X_{t_{k+1}})\Delta t + \sigma(X_{t_k})(W_{t_{k+1}} - W_{t_k}) + \frac{1}{2}\sigma\prime(X_{t_k})\sigma(X_{t_k})((W_{t_{k+1}} - W_{t_k})^2 - \Delta t)$$

1.2 Application to the Heston model and the log normal model for the variance diffusion. Conditions for ensuring the positivity of the variance process while diffusing

1.2.1 Case Heston

In this model the dynamics of the variance is the following:

$$dV_t = \kappa(\theta - V_t)dt + \eta\sqrt{V_t}dW_t$$

In relationship with the previous section we have that

$$b(V_t) = \kappa(\theta - V_t)$$

And

$$\sigma(V_t) = \eta\sqrt{V_t}$$

The Implicit Milstein is then

$$V_{t_{k+1}} = V_{t_k} + \kappa(\theta - V_{t_{k+1}})\Delta t + \eta\sqrt{V_{t_k}}\Delta W + \frac{1}{4}\eta^2(\Delta W^2 - \Delta t) \tag{3}$$

Our aim is that the process keeps positive whil we are diffusing. We will find conditions on the model parameters that will ensure this constraint.

From equation **(3)** we can write:

$$\begin{aligned} V_{t_{k+1}} &= \frac{V_{t_k} + (\kappa\theta - \frac{1}{4}\eta^2)\Delta t + \eta\sqrt{V_{t_k}}\Delta W + \frac{1}{4}\eta^2\Delta W^2}{1 + \kappa\Delta t} \\ &= \frac{N(\Delta W)}{D} \end{aligned} \tag{4}$$

The positivity of $V_{t_{k+1}}$ only depends on the numerator of equation **(4)**. We express the numerator of equation **(4)** as a function of the brownian motion increase ΔW, as it is the only risky term, uncontrollable, that can lead $V_{t_{k+1}}$ to negative values.
Let define the following function

$$g(\Delta W) := \eta\sqrt{V_{t_k}}\Delta W + \frac{1}{4}\eta^2\Delta W^2$$

We have

$$N \geq V_{t_k} + (\kappa\theta - \frac{1}{4}\eta^2)\Delta t + \min_{\Delta W} g(\Delta W)$$

And since

$$\min_{\Delta W} g(\Delta W) = -V_{t_k}$$

We get

$$N \geq (\kappa\theta - \frac{1}{4}\eta^2)\Delta t$$

$(\kappa\theta - \frac{1}{4}\eta^2)\Delta t$ is therefore a lower bound of N. This lower bound is reached almost certainly, as well as g reaches it's minimum. The brownian increase actually reaches any value in $\mathbb{R}$ with non null probability.
Therefore $(\kappa\theta - \frac{1}{4}\eta^2)\Delta t$ is exactly the function N minimum. And finally we get that a necessary and sufficient condition for keeping the variance process positive is that

$$\kappa\theta - \frac{1}{4}\eta^2 \geq 0 \tag{5}$$

This condition is less restrictive than the condition for the continuous scheme, which is:

$$\kappa\theta - \frac{1}{2}\eta^2 \geq 0$$

This means that even if the condition is not satisfied for continuous process, one can still have positive variance within implicit Milshtein scheme, as long as we respect **(5)**

1.2.2 Case Log normal

The variance dynamics is given by the following equation

$$dV_t = \kappa(\theta - V_t)dt + \eta V_t dW_t$$

The implicit Milstein scheme gives:

$$V_{t_{k+1}} = V_{t_k} + \kappa(\theta - V_{t_{k+1}})\Delta t + \eta V_{t_k}\Delta W + \frac{1}{2}\eta^2 V_{t_k}(\Delta W^2 - \Delta t)$$

And

$$\begin{aligned} V_{t_{k+1}} &= \frac{V_{t_k} + (\kappa\theta - \frac{1}{2}\eta^2 V_{t_k})\Delta t + \eta V_{t_k}\Delta W + \frac{1}{2}\eta^2 V_{t_k}\Delta W^2}{1 + \kappa\Delta t} \\ &= \frac{N(\Delta W)}{D} \end{aligned}$$

Applying the same reasoning as in Heston case gives

$$N_{min} = \kappa\theta\Delta t + \frac{1}{2}V_{t_k}(1 - \eta^2\Delta t)$$

We thus get a sufficient (but not necessary) condition for the variance positivity in the log normal case:

$$\eta^2\Delta t \leq 1$$

2 IJK Scheme for the logarithm of the stock underlying

Let consider the following diffusion

$$\begin{aligned}\frac{dS_t}{S_t} &= \mu(t)dt + V_t^p dW_t \\ dV_t &= b(V_t)dt + \sigma(V_t)dZ_t \\ d< W, Z >_t &= \rho dt\end{aligned}$$

typically $p = \frac{1}{2}$

We write itô on the logarithm of S, and we get:

$$\ln S_{t_{k+1}} = \ln S_{t_k} + \int_{t_k}^{t_{k+1}} \mu(t)dt - \frac{1}{2}\int_{t_k}^{t_{k+1}} V_t^{2p} dt + \int_{t_k}^{t_{k+1}} V_t^p dW_t$$

The terms of the previous equation are approximated as following:

$$\int_{t_k}^{t_{k+1}} \mu(t)dt \simeq \mu(t_k)\Delta t$$

$$\int_{t_k}^{t_{k+1}} V_t^{2p} dt \simeq \frac{1}{2}(V_{t_{k+1}}^{2p} + V_{t_k}^{2p})\Delta t$$

The originality of the IJK scheme lies in the way the last term ($\int_{t_k}^{t_{k+1}} V_t^p dW_t$) is approximated. Actually we will take into account the fact that when the correlation between the underlying and its variance is not null, then the diffusion of the variance's brownian motion, between two time steps, has an impact on the stock. This impact will be catched.

Cholesky decomposition of the covariance matrix gives:

$$W_t = \rho Z_t + \sqrt{(1-\rho^2)} Z_t^{\perp}$$

where Z and $Z^{\perp}$ are two independant brownian motions. We can thus write

$$\begin{aligned}\int_{t_k}^{t_{k+1}} V_t^p dW_t &= \rho \int_{t_k}^{t_{k+1}} V_t^p dZ_t + \sqrt{(1-\rho^2)} \int_{t_k}^{t_{k+1}} V_t^p Z_t^{\perp} \\ &\simeq \rho V_{t_k}^p \Delta Z + \frac{1}{2}\sqrt{(1-\rho^2)}(V_{t_{k+1}}^p + V_{t_k}^p)\Delta Z^{\perp}\end{aligned}$$

We could stop at this order for writing our scheme. However it is shown in [1] pp 23-24, that in expectation the error between the left hand term and its approximation (right hand term) is equal to

$$\frac{1}{2}\rho p V_{t_k}^p \sigma(V_{t_k})(\Delta Z^2 - \Delta t)$$

We add this correction term and we finally get the IJK scheme:

$$\begin{aligned}\ln S_{t_{k+1}} = \ln S_{t_k} + \mu(t_k)\Delta t - \frac{1}{4}(V_{t_{k+1}}^{2p} + V_{t_k}^{2p})\Delta t + \rho V_{t_k}^p \Delta Z + \frac{1}{2}\sqrt{(1-\rho^2)}(V_{t_{k+1}}^p + V_{t_k}^p)\Delta Z^{\perp} + \\ \frac{1}{2}\rho p V_{t_k}^p \sigma(V_{t_k})(\Delta Z^2 - \Delta t)\end{aligned}$$

That can be rewritten as follow

$$\begin{aligned}\ln S_{t_{k+1}} = \ln S_{t_k} + \mu(t_k)\Delta t - \frac{1}{4}(V_{t_{k+1}}^{2p} + V_{t_k}^{2p})\Delta t + \rho V_{t_k}^p \Delta Z + \frac{1}{2}(V_{t_{k+1}}^p + V_{t_k}^p)(\Delta W - \rho\Delta Z) + \\ \frac{1}{2}\rho p V_{t_k}^p \sigma(V_{t_k})(\Delta Z^2 - \Delta t)\end{aligned}$$

References

[1] Jackel P. and Kahl C.: May 2006, Fast strong approximation Monte-Carlo schemes for stochastic volatility models

[2] Gunter M., Kahl C., Rossberg T.: September 2004, Structure preserving stochastic integration schemes in interest rate derivative modeling