Realized Volatility of Volatility For a fixed maturity*

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*This note is entirely inspired by [1]

1 Preliminaries

Let remind the spot and variance diffusions in double log normal model:

$$\begin{aligned}\frac{dF_t}{F_t} &= \sqrt{V_t}dW_t^F \\ dV_t &= \kappa(\hat{V}_t - V_t)dt + \eta_1 V_t dW_t^{SD} \\ d\hat{V}_t &= c(\hat{V}_\infty - \hat{V}_t)dt + \eta_2 \hat{V}_t dW_t^{LD}\end{aligned}$$

with

$$\begin{aligned}d < W^F, W^{SD} >_t &= \rho_{SD}dt \\ d < W^F, W^{LD} >_t &= \rho_{LD}dt \\ d < W^{SD}, W^{LD} >_t &= \rho dt\end{aligned}$$

At a given time t, the Variance Swap starting at t with maturity T is given by

$$VS(t,T) = \mathbb{E}_t\left[\frac{1}{T-t}\int_t^T V_s ds\right]$$

At a given time t, a Variance Swap started at 0 with maturity T worths

$$\begin{aligned}VS(0,T)_t &= \frac{1}{T}\int_0^t V_s ds + \mathbb{E}_t\left[\frac{1}{T}\int_t^T V_s ds\right] \\ &= \frac{1}{T}\int_0^t V_s ds + \frac{T-t}{T}VS(t,T)\end{aligned}$$

With a fixed maturity T, applying a dynamic with respect to t only, we get

$$\frac{dVS(0,T)_t}{VS(0,T)_t} = (...)dt + \frac{(T-t)}{T}\frac{dVS(t,T)}{VS(0,T)_t} \tag{1}$$

2 Realized Volatility of the Square Root of the variance Swap in double log normal Model

In double log normal model, Variance Swap is given by

$$VS(t,T) = \frac{1}{\kappa(T-t)}\left(V_t - \frac{\kappa(\hat{V}_t - V_\infty)}{\kappa - c} - V_\infty\right)(1 - e^{-\kappa(T-t)}) + \frac{\kappa(\hat{V}_t - V_\infty)}{c(T-t)(\kappa-c)}(1 - e^{-c(T-t)}) + V_\infty$$

Then, always with T fixed, we have

$$dVS(t,T) = (...)dt + \frac{\eta_1 V_t}{\kappa(T-t)}(1-e^{-\kappa(T-t)})dW_t^{SD} + \frac{\kappa\eta_2\hat{V}_t}{(\kappa-c)(T-t)}\left(\frac{(1-e^{-c(T-t)})}{c} - \frac{(1-e^{-\kappa(T-t)})}{\kappa}\right)dW_t^{LD}$$

Using **(1)**, we get

$$\begin{aligned}\frac{dVS(0,T)_t}{VS(0,T)_t} &= (...)dt + \frac{\eta_1}{\kappa T}\frac{V_t}{VS(0,T)_t}(1 - e^{-\kappa(T-t)})dW_t^{SD} \\ &\quad + \frac{\kappa\eta_2}{(\kappa-c)T}\frac{\hat{V}_t}{VS(0,T)_t}\left(\frac{(1-e^{-c(T-t)})}{c} - \frac{(1-e^{-\kappa(T-t)})}{\kappa}\right)dW_t^{LD} \\ &= (...)dt + \sqrt{VaVar_t}dW_t\end{aligned}$$

The realized variance of $VS(0,T)$ is then given by

$$\begin{aligned}
realizedVar(VS(0,T)) &= \mathbb{E}_0\left[\frac{1}{T}\int_0^T VaVar_t dt\right] \\
&= \frac{\eta_1^2}{\kappa^2T^3}\int_0^T (1-e^{-\kappa(T-t)})^2\mathbb{E}_0\left[\frac{V_t^2}{VS(0,T)_t^2}\right]dt \\
&\quad +\frac{\eta_2^2\kappa^2}{(\kappa-c)^2T^3}\int_0^T\left(\frac{(1-e^{-c(T-t)})}{c}-\frac{(1-e^{-\kappa(T-t)})}{\kappa}\right)^2\mathbb{E}_0\left[\frac{\hat{V}_t^2}{VS(0,T)_t^2}\right]dt \\
&\quad +\frac{2\rho\eta_1\eta_2}{(\kappa-c)T^3}\int_0^T(1-e^{-\kappa(T-t)})\left(\frac{(1-e^{-c(T-t)})}{c}-\frac{(1-e^{-\kappa(T-t)})}{\kappa}\right)\mathbb{E}_0\left[\frac{V_t\hat{V}_t}{VS(0,T)_t^2}\right]dt
\end{aligned}$$

We make the approximations that the expressions into parenthesis are close to 1. And we have

$$\begin{aligned}
realizedVaVar(VS(0,T)) &= \frac{\eta_1^2}{\kappa^2T^3}\int_0^T(1-e^{-\kappa(T-t)})^2dt+\frac{\eta_2^2\kappa^2}{(\kappa-c)^2T^3}\int_0^T\left(\frac{(1-e^{-c(T-t)})}{c}-\frac{(1-e^{-\kappa(T-t)})}{\kappa}\right)^2dt \\
&\quad +\frac{2\rho\eta_1\eta_2}{(\kappa-c)T^3}\int_0^T(1-e^{-\kappa(T-t)})\left(\frac{(1-e^{-c(T-t)})}{c}-\frac{(1-e^{-\kappa(T-t)})}{\kappa}\right)dt
\end{aligned}$$

And finally, we calculate the realized volatility of the Variance Swap Square root, through the formula:

$$realizedVoVol(0,T)=\frac{1}{2}\sqrt{realizedVar(VS(0,T))}$$

We remind that for a rolling Variance Swap of maturity θ, the realized volatility of the Variance Swap square root is given by

$$realzdRollVoVol(\theta)=\frac{1}{2}\sqrt{\frac{\eta_1^2}{\kappa^2\theta^2}(1-e^{-\kappa\theta})^2+\frac{\kappa^2\eta_2^2}{\theta^2(\kappa-c)^2}\left(\frac{(1-e^{-c\theta})}{c}-\frac{(1-e^{-\kappa\theta})}{\kappa}\right)^2+\frac{2\rho\eta_1\eta_2}{\theta^2(\kappa-c)}(1-e^{-\kappa\theta})\left(\frac{(1-e^{-c\theta})}{c}-\frac{(1-e^{-\kappa\theta})}{\kappa}\right)}$$

Remarks

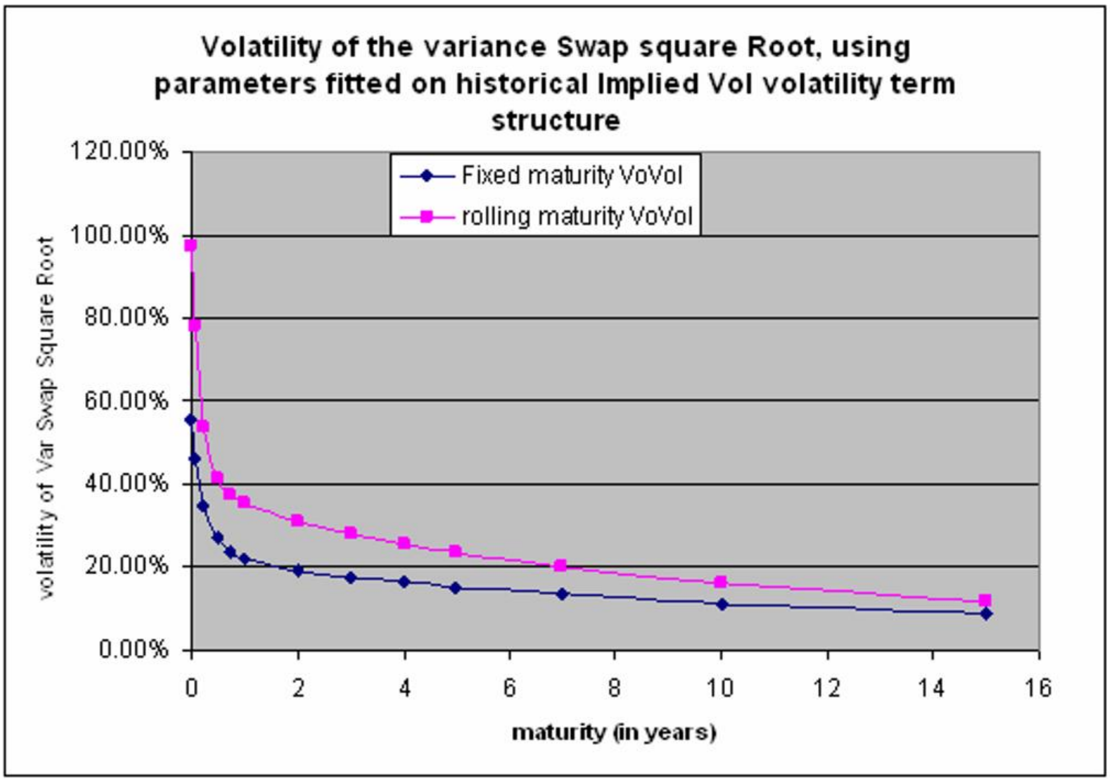
- The realized volatility of volatility is decreasing with the maturity T;
- As $T\longrightarrow 0^+$, $realizedVoVol(0,T)\longrightarrow\frac{\eta_1}{2\sqrt{3}}$, which is it's maximum value;
- As $T\longrightarrow +\infty$, $realizedVoVol(0,T)\longrightarrow 0^+$;
- As $\theta\longrightarrow 0^+$, $realzdRollVoVol(\theta)\longrightarrow\frac{\eta_1}{2}$, which is it's maximum value;
- As $\theta\longrightarrow +\infty$, $realzdRollVoVol(\theta)\longrightarrow 0^+$;
- It seems like we have the inequality: $realzdRollVoVol(\theta)\geq realizedVoVol(0,\theta)$.

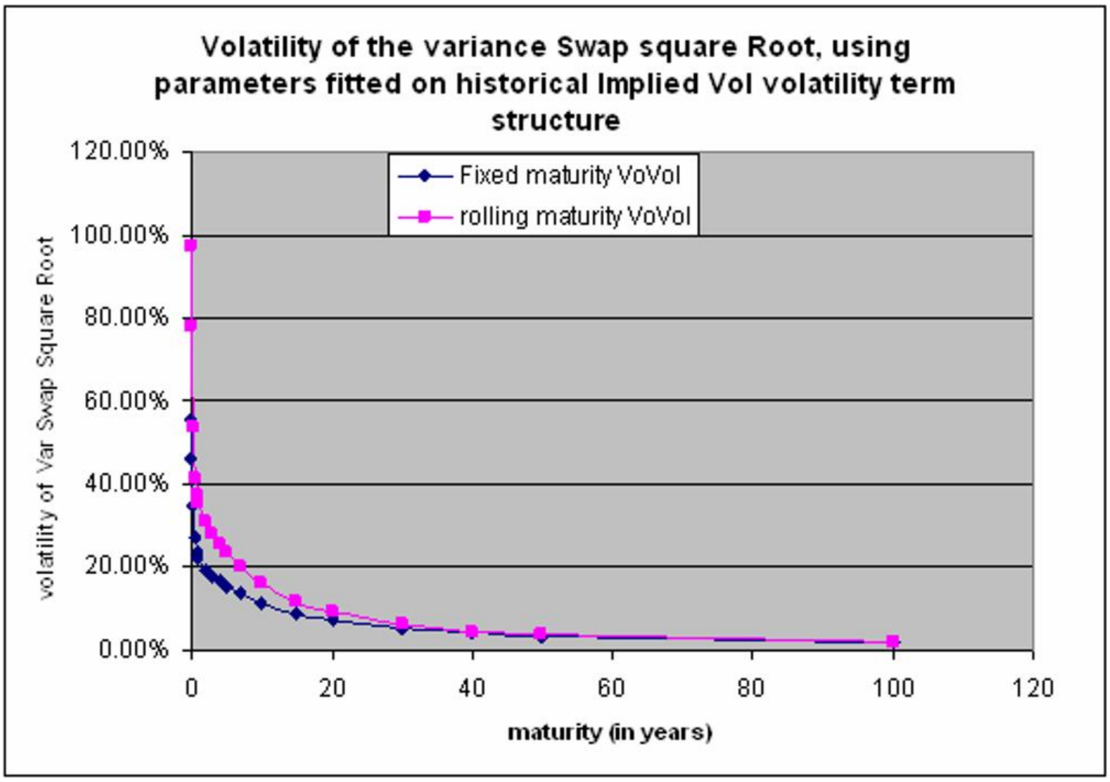
3 Numerical Tests

We focus on the STOXX 50E. First we use a set of parameters obtained by fitting the term structure of the historical rolling implied volatility's volatility, and the term structure of the correlation "Spot-Rolling Variance Swap" simultaneously :

	STOXX50E
κ	763.26%
c	20.35%
η_S	194.39%
η_L	73.89%
ρ	28.43%

We get the following graphics:

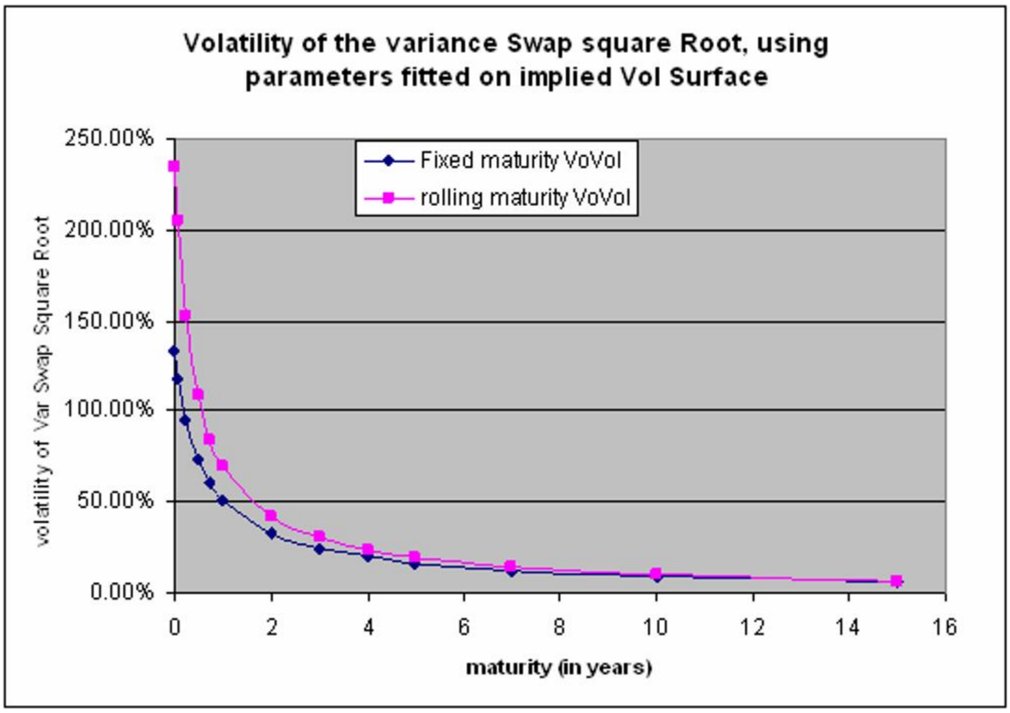




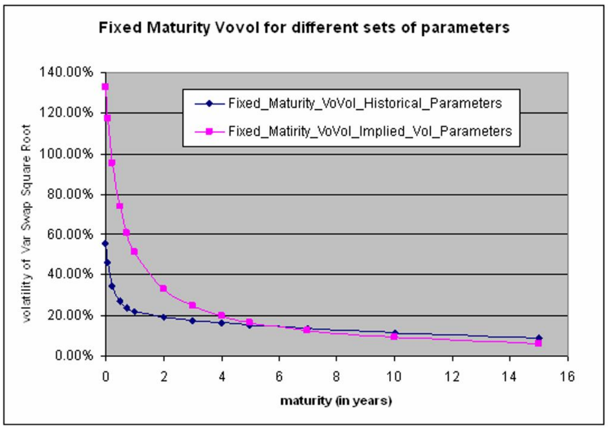
Then, we use a set of parameters got by calibrating the implied volatility surface, through a Monte Carlo pricer:

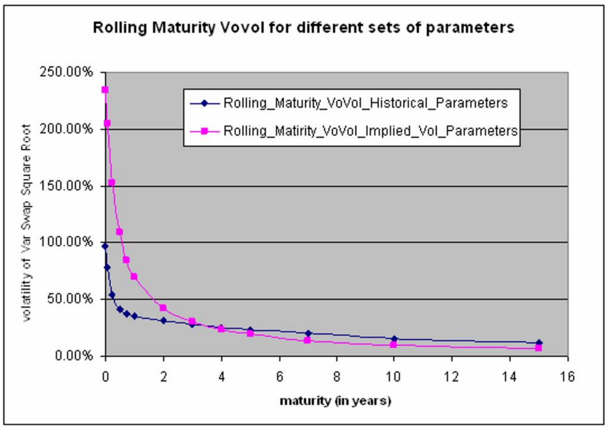
	STOXX50E
κ	446.00%
c	74.86%
η_S	455.96%
η_L	87.28%
ρ	63.59%

We get the following term structure



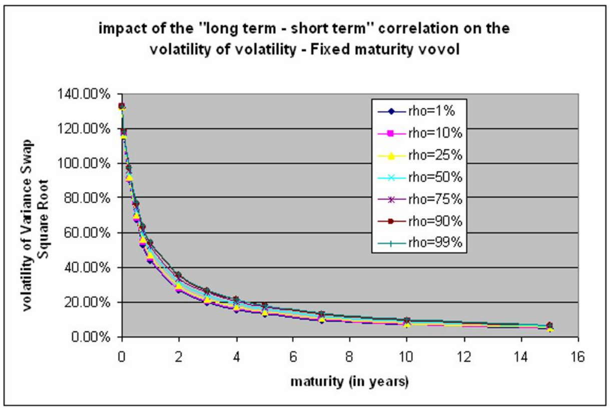
The comparison between the two sets of parameters is given on the next figure:

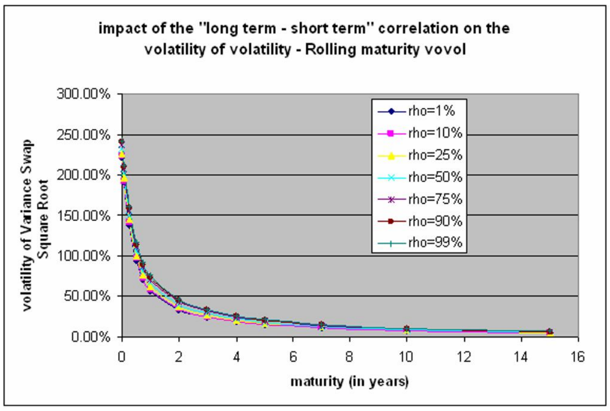




Impact of the "long term - short term" Correlation: ρ

In order to study the impact of the "long term - short term" Correlation on the term of volatility of volatility, we show the figures below:





It appears that ρ could have some impact around the mid term (1, 2, 3 years): about 15%. But It's impact is negligible around the short and the long term period.

References

[1] KOUOKAP YOUMBI D.: October 2008, 2 Factors Stochastic Volatility Models: Formula for the Variance Swap variance and asymptotic formula for the At The Money Forward (ATMF) Skew, *Internship Report*