Ecole Polytechnique
Master 2 Probabilities And Finance
(Paris 6)

HSBC FRANCE
Quantitative Research,Structured Equity

Internship Report

# 2 Factors Stochastic Volatility Models: Formula for the Variance Swap variance and asymptotic formula for the At The Money Forward (ATMF) Skew

By

Didier KOUOKAP YOUMBI

Supervisor

Pascal DELANOE

April $15^{th}$ 2008 - October $15^{th}$ 2008

# Contents

# Abstract

In this report, we provide formula for the zero-strike Variance Swap variance. We also provide asymptotic formula for the ATMF skew, in a context of three particular 2 factors Stochastic Volatility Models. As application, we use these formula to calibrate the mean reversion parameters on the term structure of Variance Swap variance.

Most products, especially those for which a static replication is possible, are priced using volatily (local or stochastic) calibrated on market vanilla options (Puts and Calls). However, some exotic products like Napoleon are very sensitive to some first or second order features (forward skew or/and volatility of volatility) others than the vanilla prices. For pricing Napoleon for instance, Bergomi illustrates in **[4]** that we need to be well calibrated on the volatility of volatility.

What we do in this report is to express, analytically, variance of Variance Swap as a function of our models parameters. Then, we use these formula to fit term structure of historical variance of implied variance*.

The fit is done through Powell algorithm combined with conjugated gradient. Calibration is very fast (less than 3 minutes for a fit), but we need to find the accurate initial point that make the algorithm converge. We obtain good results enough, and parameters we get are consistent with their intuitive meanings.

*Implied Variance being the square of the implied volatility.

# Remerciements

Je remercie Nicolas GRANDCHAMP DES RAUX, qui m'a accepté au sein de son équipe, et pour son suivi, ainsi que ses conseils continus pour la réussite de ce projet de volatilité stochastique 2 facteurs.

Je remercie tout particulièrement Pascal et Alexandre pour leur encadrement quotidien, leur grande disponibilité ainsi que la patience dont ils ont fait preuve en mon égard.

Je remercie enfin tous les autres membres de l'équipe de recherche Dérivés Actions, pour m'avoir gentiment accueilli au sein de l'équipe, facilitant ainsi mon intégration.

Henri, merci pour ces magnifiques parties de *Henri Poker Tour*, et vivement que ça recommence.

Je n'oublie pas Nicolas (Huth), Hamid et Olivier.

Didier

# Introduction

In order to replicate some exotic products such as Cliquet or Napoleon, we need to fit the term structure of volatility of volatility, and the forward smile. There is no liquidity on these products. However, we know that the inverse of mean reversion parameter is homogeneous to the time scale, and moreover, volatility of volatility strongly depends on mean reversion parameters. Thus we can fit the mean reversion parameters on the term structure of historical volatility volatility. But since volatility is not a tradable product, we will fit historical Variance Swap variance.

In the first section we present the three 2 factors stochastic volatility models on which we work: we write the models dynamics and we comment specifications of each model.

In section 2, for each of our three models, we express Variance Swap Variance as a function of model parameters.

In section 3 we provide asymptotic formula for the ATMF Skew.

Section 4 deals with the computational results.

# 1 Models Presentation

For the sake of simplification, we suppose a zero interest rate. Or if one supposes that the underlying is the forward, then we will have no drift term in the diffusion equation, under the risk neutral Probability[†].

## 1.1 2 Factors Balland's Model

### 1.1.1 Model dynamic

In this Model, the underlying diffuses as follow:

$$\begin{aligned} \frac{dF_t}{F_t} &= \sqrt{V_t} dW_t^F \\ \sqrt{V_t} &= \sqrt{V_0} e^{(Z_t^{SD} + Z_t^{LD} - \frac{1}{2} var(Z_t^{SD} + Z_t^{LD}))} \end{aligned}$$

With

$$\begin{aligned} Z_t^{SD} &= \gamma_{SD} \int_0^t e^{\lambda_{SD}(s-t)} dW_s^{SD} \\ Z_t^{LD} &= \gamma_{LD} \int_0^t e^{\lambda_{LD}(s-t)} dW_s^{LD} \end{aligned}$$

And

$$\begin{aligned} d < W^F, W^{SD} >_t &= \rho_{SD} dt \\ d < W^F, W^{LD} >_t &= \rho_{LD} dt \\ d < W^{SD}, W^{LD} >_t &= \rho dt \end{aligned}$$

- $\frac{1}{\lambda_{LD}}$ characterizes the long period duration and $\frac{1}{\lambda_{SD}}$ the short period duration.
- $Z^{SD}$ and $Z^{LD}$ are Ornstein Uhlenbeck processes with mean reversion $\lambda$ and 0 as long term variance. In fact, we have

$$dZ_t^i = -\lambda_i Z_t^i dt + \gamma_i dW_t^i$$
$$i \in \{\mathrm{SD}, \mathrm{LD}\}$$

- There should be some relations between the parameters:

$$\frac{\gamma_{SD}^2}{\lambda_{SD}} = \mathbf{O(1)} \quad (1)$$
$$\lambda_{SD} >> 1 \quad (2)$$
and
$$\frac{\lambda_{SD}}{\lambda_{LD}} >> 1 \quad (3)$$

- Relation **(1)** ensures an equilibrium between term in $dt$ and term in $dW_t$ within the diffusion equation.
- Equations **(2)** and **(3)** are due to time scales. When **(3)** is satisfied, then there is separation between short and long term. This implies:
  - $\rho \simeq 0$;
  - We can fit the long term and short term data separately;

[†]The forward is a martingale under the risk neutral probability

### 1.1.2 Balland To SABR

In term of SDE[‡], we can rewrite the diffusion as follow:

$$\begin{aligned}
\frac{dF_t}{F_t} &= \sqrt{V}_t dW_t^F \\
\frac{dV_t}{V_t} &= \{2\gamma_{SD}^2 + 2\gamma_{LD}^2 + 4\rho\gamma_{SD}\gamma_{LD} - \gamma_{SD}^2 e^{-2\lambda_{SD}t} - \gamma_{LD}^2 e^{-2\lambda_{LD}t} - \\
&\quad 2\rho\gamma_{SD}\gamma_{LD}e^{-(\lambda_{SD}+\lambda_{LD})t} - 2\lambda_{SD}Z_t^{SD} - 2\lambda_{LD}Z_t^{LD}\}dt + 2\gamma_{SD}dW_t^{SD} + 2\gamma_{LD}dW_t^{LD} \quad (4) \\
dZ_t^{SD} &= -\lambda_{SD}Z_t^{SD}dt + \gamma_{SD}dW_t^{SD} \\
dZ_t^{LD} &= -\lambda_{LD}Z_t^{LD}dt + \gamma_{LD}dW_t^{LD}
\end{aligned}$$

With

$$\begin{aligned}
d < W^F, W^{SD} >_t &= \rho_{SD}dt \\
d < W^F, W^{LD} >_t &= \rho_{LD}dt \\
d < W^{SD}, W^{LD} >_t &= \rho dt
\end{aligned}$$

In particular for Balland 1 factor (when $\lambda_{LD} = 0$ and $\gamma_{LD} = 0$), for $\lambda_{SD} = 0$, equation **(4)** gives:

$$\frac{dV_t}{V_t} = \gamma^2 dt + 2\gamma dW_t^{SD}$$

That is equivalent to

$$\frac{d\sqrt{V_t}}{\sqrt{V_t}} = \gamma dW_t^{SD}$$

As a consequence, SABR model (with $\beta = 1$ ) can be seen as a particular case of 1 factor Balland model. Since we already know how to handle SABR model, we can use this to have a first guess on some Balland's parameters. Or we can compare some Balland's parameters to those of SABR with $\beta = 1$, in order to see how good is our Balland's calibration.

## 1.2 Double Lognormal model (2 Factors Gatheral's Model)

The underlying's dynamic in this model is given by:

$$\begin{aligned}
\frac{dF_t}{F_t} &= \sqrt{V_t}dW_t^F \\
dV_t &= \kappa(\hat{V}_t - V_t)dt + \eta_1 V_t dW_t^{SD} \\
d\hat{V}_t &= c(\hat{V}_\infty - \hat{V}_t)dt + \eta_2\hat{V}_t dW_t^{LD}
\end{aligned}$$

with

$$\begin{aligned}
d < W^F, W^{SD} >_t &= \rho_{SD}dt \\
d < W^F, W^{LD} >_t &= \rho_{LD}dt \\
d < W^{SD}, W^{LD} >_t &= \rho dt
\end{aligned}$$

The mean reversion parameters here are $\kappa$ end $c$. They have the same meaning as $\lambda_{SD}$ and $\lambda_{LD}$ in Balland's model respectively. $\eta_1$ and $\eta_2$ also have the same meaning as $\gamma_{SD}$ and $\gamma_{LD}$ respectively.

Variance here is a mean reversion process that reverts toward a process, that itself reverts toward a long term level ($\hat{V}_\infty$).

For times around 0, the variance reverts toward $\hat{V}_0$. Then $\hat{V}_0$ can be seen as the short term mean level of the variance. Finally, $V_0$, $\hat{V}_0$ and $\hat{V}_\infty$ morally impose an intuitive term structure of the variance.

[‡]Stochastic Differential Equation

Gatheral's model differs from Balland in many ways:

- In Balland, Variance is lognormal while in Gatheral it's not. That may make calculus easier in Balland.
- in Gatheral,Variance may be less sensitive to the long term parameters than in Balland

### 1.3 Double Heston model

In double Heston Model, the underlying dynamic is:

$$\begin{aligned} \frac{dF_t}{F_t} &= \sqrt{V_t} dW_t^F \\ dV_t &= \kappa(\hat{V}_t - V_t)dt + \eta_1 \sqrt{V_t} dW_t^{SD} \\ d\hat{V}_t &= c(\hat{V}_\infty - \hat{V}_t)dt + \eta_2 \sqrt{\hat{V}_t} dW_t^{LD} \end{aligned}$$

with

$$\begin{aligned} d < W^F, W^{SD} >_t &= \rho_{SD} dt \\ d < W^F, W^{LD} >_t &= \rho_{LD} dt \\ d < W^{SD}, W^{LD} >_t &= \rho dt \end{aligned}$$

The variance here diffuses as a CIR[§] model for the interest rate. And it reverts toward a process that follows, itself, a CIR process and reverts toward a long duration level.

Both Gatheral and Double Heston models are particular cases of a global class of models called double CEV, which can be written as follow:

$$\begin{aligned} \frac{dF_t}{F_t} &= \sqrt{V_t} dW_t^F \\ dV_t &= \kappa(\hat{V}_t - V_t)dt + \eta_1 V_t^\alpha dW_t^{SD} \\ d\hat{V}_t &= c(\hat{V}_\infty - \hat{V}_t)dt + \eta_2 \hat{V}_t^\beta dW_t^{LD} \end{aligned}$$

with

$$\begin{aligned} d < W^F, W^{SD} >_t &= \rho_{SD} dt \\ d < W^F, W^{LD} >_t &= \rho_{LD} dt \\ d < W^{SD}, W^{LD} >_t &= \rho dt \end{aligned}$$

And

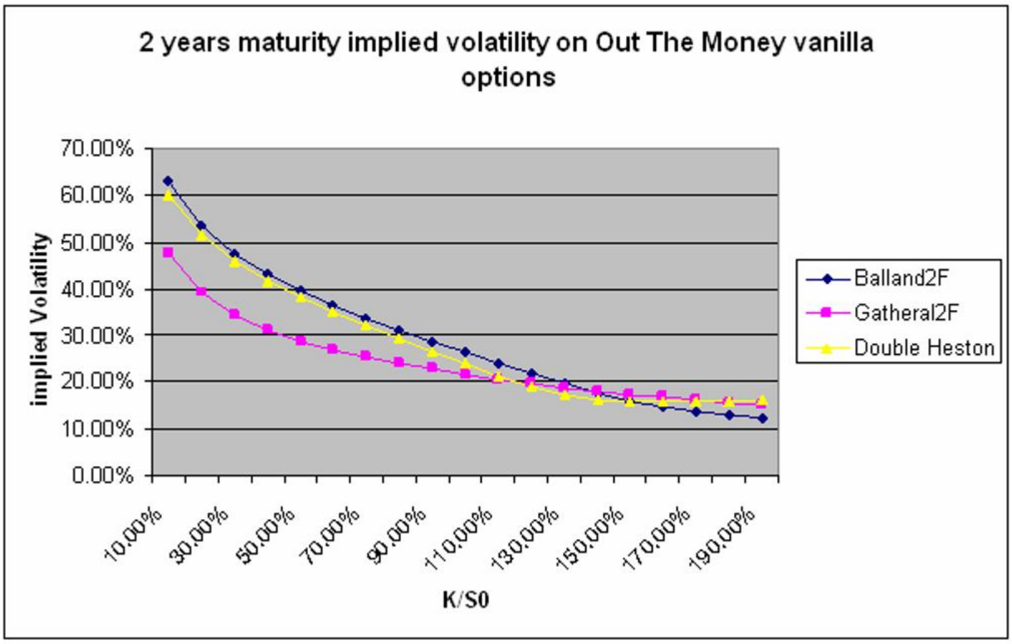
$$\alpha, \beta \in [\frac{1}{2}; 1].$$

For the same set of parameters:

$\{\kappa = \lambda_{SD} = 740.17\%; c = \lambda_{LD} = 10.48\%; \eta_1 = \gamma_{SD} = 272\%; \eta_2 = \gamma_{LD} = 33.2\%; \rho_S = -87.59\%;$
$\rho_L = -50.62\%; \rho_{SL} = 2.73\%; V_0 = 5.78\%; \hat{V}_0 = 5.20\%; V_\infty = 6\%\}$

We draw on the next figure the implied volatility on a 2 years maturity out of the money vanilla options, for each of our three models.

[§] Cox-Ingersoll-Ross




We observe that Balland 2 Factors seems to be the most expensive for ($I = \frac{K}{S}$) under 100% and the Cheapest for $I$ over 100%.
Double Heston prices stay between Balland and Gatheral.

# 2 Calculation of zero-Strike rolling Variance Swap's variance

In this section, for each of our three models, we will calculate the variance of a zero-strike variance swap.

## 2.1 Preliminaries

Let us recall the following well-known definitions and results.

- Given an underlying that diffuses as

$$\frac{dS_t}{S_t} = \mu_t dt + \sigma_t dW_t$$

The realized variance on the underlying $S$ between 0 and $t$ is given by

$$\frac{1}{t}\int_0^t \sigma_s^2 ds$$

Writing Itô's lemma on the logarithm of S, we get

$$d\ln S_t = \frac{dS_t}{S_t} - \frac{1}{2}\sigma_t^2 dt$$

Thus, the martingale part of $\ln S$ is $\int \sigma_s dW_s$. And the quadratic variation of $\ln S$ is

$$< \ln S, \ln S >_t = \int_0^t \sigma_s^2 ds$$

- The zero-strike forward Variance Swap starting at time t, with Maturity $\theta$ is given by

$$\begin{aligned} VS(t,t+\theta) &= \mathrm{E}_t(\frac{1}{\theta}\int_t^{t+\theta} \sigma_s^2 ds) \\ &= \frac{1}{\theta}\int_t^{t+\theta} E_t(\sigma_s^2) ds \\ &= \frac{1}{\theta}\int_t^{t+\theta} V_{t,s} ds \end{aligned} \tag{5}$$

where

$$\mathrm{E}_t(.) = \mathrm{E}(./\mathcal{F}_t)$$

And

$$V_{t,s} = E_t(\sigma_s^2) \ ; \ 0 \leq t \leq s$$

By differentiating the equation **(5)**, we get:

$$dVS(t,t+\theta) = \frac{1}{\theta}\left\{(V_{t,t+\theta} - V_{t,t})dt + \int_t^{t+\theta} dV_{t,s} ds\right\} \tag{6}$$

## 2.2 Zero-strike Variance Swap variance in 2 Factors Balland's model

### 2.2.1 Calculation of forward variance in Balland's: $V_{t,T} = \mathbf{E}_t(V_T)\ \forall\ 0 \leq t \leq T$

We remind that

$$V_T = V_0 e^{\left\{2(Z_T^{SD}+Z_T^{LD})-var(Z_T^{SD}+Z_T^{LD})\right\}}$$

For $i \in \{\text{SD ; LD}\}$, we write

$$Z_T^i = e^{\lambda_i(t-T)} Z_t^i + X_{t,T}^i$$

Where

$$X_{t,T}^i = \gamma_i \int_t^T e^{\lambda_i(s-T)} dW_s^i$$

Using the fact that $(X_{t,T}^{SD}, X_{t,T}^{LD})$ is gaussian, we get

$$V_{t,T} = V_0 e^{\left\{2[e^{\lambda_{SD}(t-T)}Z_t^{SD}+e^{\lambda_{LD}(t-T)}Z_t^{LD}]+2Var(X_{t,T}^{SD}+X_{t,T}^{LD})-Var(Z_T^{SD}+Z_T^{LD})\right\}}$$

We write Itô on $V_{t,T}$. We get (T is fixed)

$$\begin{aligned}\frac{dV_{t,T}}{V_{t,T}} &= \frac{\partial}{\partial t}\left(\ln\{\frac{V_{t,T}}{V_0}\}\right)dt + 2e^{\lambda_{SD}(t-T)}dZ_t^{SD} + 2e^{\lambda_{LD}(t-T)}dZ_t^{LD} + 2e^{2\lambda_{SD}(t-T)}d<Z^{SD},Z^{SD}>_t + \\ & 2e^{2\lambda_{LD}(t-T)}d<Z^{LD},Z^{LD}>_t + 4e^{(\lambda_{SD}+\lambda_{LD})(t-T)}d<Z^{SD},Z^{LD}>_t\end{aligned}$$

And since

$$Var(Z_t^{SD}+Z_t^{LD}) = \frac{\gamma_{SD}^2}{2\lambda_{SD}}(1-e^{-2\lambda_{SD}t}) + \frac{\gamma_{LD}^2}{2\lambda_{LD}}(1-e^{-2\lambda_{LD}t}) + 2\frac{\rho\gamma_{SD}\gamma_{LD}}{\lambda_{SD}+\lambda_{LD}}\left(1-e^{-(\lambda_{SD}+\lambda_{LD})t}\right)$$

And

$$Var(X_{t,T}^{SD}+X_{t,T}^{LD}) = \frac{\gamma_{SD}^2}{2\lambda_{SD}}(1-e^{-2\lambda_{SD}(T-t)}) + \frac{\gamma_{LD}^2}{2\lambda_{LD}}(1-e^{-2\lambda_{LD}(T-t)}) + 2\frac{\rho\gamma_{SD}\gamma_{LD}}{\lambda_{SD}+\lambda_{LD}}\left(1-e^{-(\lambda_{SD}+\lambda_{LD})(T-t)}\right)$$

We finally derive the following expression:

$$\frac{dV_{t,T}}{V_{t,T}} = 2\gamma_{SD}e^{-\lambda_{SD}(T-t)}dW_t^{SD} + 2\gamma_{LD}e^{-\lambda_{LD}(T-t)}dW_t^{LD} \tag{7}$$

We find that forward variance is a local martingale under the risk neutral probability.

### 2.2.2 Calculation of the Variance Swap's realized variance in Balland's

We substitute **(7)** in **(6)**. We obtain

$$dVS(t,t+\theta) = \frac{1}{\theta}\left\{(V_{t,t+\theta} - V_{t,t})dt + \left(\int_t^{t+\theta} 2\gamma_{SD}V_{t,s}e^{\lambda_{SD}(s-t)}ds\right)dW_t^{SD} + \left(\int_t^{t+\theta} 2\gamma_{LD}V_{t,s}e^{\lambda_{LD}(s-t)}ds\right)dW_t^{LD}\right\}$$

We then make the approximation that between t and $t+\theta$ we have $V_{t,s} \approx VS(t,t+\theta)$. It means that we assume a flat variance term structure between t and $t+\theta$. Which may be realistic if the maturity $\theta$ is short.
The approximation leads to

$$\frac{dVS(t,t+\theta)}{VS(t,t+\theta)} = \frac{1}{\theta VS(t,t+\theta)}(V_{t,t+\theta} - V_{t,t})dt + \frac{1}{\theta}\left(\frac{2\gamma_{SD}}{\lambda_{SD}}(1-e^{-\lambda_{SD}\theta}\right)dW_t^{SD} + \frac{1}{\theta}\left(\frac{2\gamma_{LD}}{\lambda_{LD}}(1-e^{-\lambda_{LD}\theta})\right)dW_t^{LD}$$

We can thus easily calculate the quadratic variation of $\ln(VS(t,t+\theta))$ (with $\theta$ fixed). The result is

$$< \ln(VS(.,.+\theta)), \ln(VS(.,.+\theta)) >_t =$$
$$\frac{t}{\theta^2}\left(\frac{4\gamma_{SD}^2}{\lambda_{SD}^2}(1-e^{-\lambda_{SD}\theta})^2 + \frac{4\gamma_{LD}^2}{\lambda_{LD}^2}(1-e^{-\lambda_{LD}\theta})^2 + \frac{8\rho\gamma_{SD}\gamma_{LD}}{\lambda_{SD}\lambda_{LD}}(1-e^{-\lambda_{SD}\theta})(1-e^{-\lambda_{LD}\theta})\right)$$

The realized variance, between 0 and t, on the rolling (constant maturity $\theta$) Variance Swap is then

$$\frac{1}{t} < \ln(VS(.,.+\theta)), \ln(VS(.,.+\theta)) >_t =$$
$$\frac{1}{\theta^2}\left(\frac{4\gamma_{SD}^2}{\lambda_{SD}^2}(1-e^{-\lambda_{SD}\theta})^2 + \frac{4\gamma_{LD}^2}{\lambda_{LD}^2}(1-e^{-\lambda_{LD}\theta})^2 + \frac{8\rho\gamma_{SD}\gamma_{LD}}{\lambda_{SD}\lambda_{LD}}(1-e^{-\lambda_{SD}\theta})(1-e^{-\lambda_{LD}\theta})\right) \quad (8)$$

We observe that realized variance on Variance Swap strongly depends on mean reversion parameters $\lambda_{SD}$ and $\lambda_{LD}$. So, catching the term structure of variance may be equivalent to obtain good enough values for $\lambda_{SD}$ and $\lambda_{LD}$

### 2.3 Zero-strike Variance Swap variance in 2 Factors Gatheral's model (Double Log normal)

#### 2.3.1 Calculation of forward variance in Gatheral's: $V_{t,T} = \mathbf{E}_t(V_T)\ \forall\ 0 \le t \le T$

We have

$$dV_t = \kappa(\hat{V}_t - V_t)dt + \eta_1 V_t dW_t^{SD}$$

Let $v_t = e^{\kappa t} V_t$. We have

$$dv_t = \kappa e^{\kappa t}\hat{V}_t dt + \eta_1 e^{\kappa t} V_t dW_t^{SD}$$

By integrating we get

$$\begin{aligned} V_t &= V_0 e^{-\kappa t} + \kappa \int_0^t e^{\kappa(s-t)}\hat{V}_s ds + \eta_1 \int_0^t e^{\kappa(s-t)} V_s dW_s^{SD} \\ &= V_0 e^{-\kappa t} + X_t + Z_t \end{aligned}$$

Where

$$X_t = \kappa \int_0^t e^{\kappa(s-t)}\hat{V}_s ds$$

And

$$Z_t = \eta_1 \int_0^t e^{\kappa(s-t)} V_s dW_s^{SD}$$

Then we have

$$\begin{aligned} V_{t,T} &= E_t(V_T) \\ &= E_t(V_0 e^{-\kappa T} + X_T + Z_T) \end{aligned} \tag{9}$$

We easily show that

$$Z_T = e^{\kappa(t-T)} \left( Z_t + \eta_1 \int_t^T e^{\kappa(s-t)} V_s dW_s^{SD} \right)$$

And

$$X_T = e^{\kappa(t-T)} \left( X_t + \kappa \int_t^T e^{\kappa(s-t)}\hat{V}_s ds \right)$$

Replacing in **(9)** we get

$$\begin{aligned} V_{t,T} &= V_0 e^{-\kappa T} + e^{\kappa(t-T)} \left( X_t + Z_t + \kappa \int_t^T e^{\kappa(s-t)} E_t(\hat{V}_s) ds \right) \\ &= V_0 e^{-\kappa T} + e^{\kappa(t-T)} \left( V_t - V_0 e^{-\kappa t} + \kappa \int_t^T e^{\kappa(s-t)} E_t(\hat{V}_s) ds \right) \end{aligned} \tag{10}$$

Where we've admitted that the process

$$t \rightarrowtail \eta_1 \int_t^T e^{\kappa(s-t)} V_s dW_s^{SD}$$

is a true martingale (we can prove this when T is fixed).

We now need to calculate $\hat{V}_{t,T} = E_t(\hat{V}_T)$, $(0 \leq t \leq T)$. For that we use the same method as we are doing now for $E_t(V_T)$. The result is:

$$\hat{V}_{t,T} = (\hat{V}_t - \hat{V}_\infty)e^{-c(T-t)} + \hat{V}_\infty$$

We replace in **(10)**, and after calculation, we get:

$$V_{t,T} = \left( V_t - \frac{\kappa(\hat{V}_t - \hat{V}_\infty)}{\kappa - c} - \hat{V}_\infty \right) e^{-\kappa(T-t)} + \frac{\kappa(\hat{V}_t - \hat{V}_\infty)}{\kappa - c} e^{-c(T-t)} + \hat{V}_\infty \tag{11}$$

### 2.3.2 Calculation of the Variance Swap realized variance in Gatheral's

We insert **(11)** in **(5)**. And after calculation we derive the following formula for Variance Swap

$$\theta VS(t, t+\theta) = \theta \hat{V}_\infty + \frac{(V_t - \frac{\kappa(\hat{V}_t - \hat{V}_\infty)}{\kappa - c} - \hat{V}_\infty)}{\kappa}(1 - e^{-\kappa\theta}) + \frac{\kappa(\hat{V}_t - \hat{V}_\infty)}{c(\kappa - c)}(1 - e^{-c\theta})$$

Now we can easily calculate the quadratic variation of the log of variance swap. We get

$$< \ln(VS(.,.+\theta)), \ln(VS(.,.+\theta)) >_t = \frac{\eta_1^2(1-e^{-\kappa\theta})^2}{\theta^2\kappa^2} \int_0^t \frac{V_s^2}{VS(s,s+\theta)^2} ds + \frac{\eta_2^2\kappa^2\left(\frac{(1-e^{-c\theta})}{c} - \frac{(1-e^{-\kappa\theta})}{\kappa}\right)^2}{\theta^2(\kappa-c)^2} \int_0^t \frac{\hat{V}_s^2}{VS(s,s+\theta)^2} ds + \frac{2\rho\eta_1\eta_2}{\theta^2(\kappa-c)}(1 - e^{-\kappa\theta}) \left(\frac{(1-e^{-c\theta})}{c} - \frac{(1-e^{-\kappa\theta})}{\kappa}\right) \int_0^t \frac{V_s\hat{V}_s}{VS(s,s+\theta)^2} ds$$

We make the approximation that between 0 and $t$, $V_s \approx VS(s, s+\theta)$ and that $\hat{V}_s \approx VS(s, s+\theta)$. That is equivalent to assume an almost flat variance term structure between 0 and t.

We then derive the realized variance on Variance Swap, between 0 and t.

$$\frac{1}{t} < \ln(VS(.,.+\theta)), \ln(VS(.,.+\theta)) >_t = \frac{\eta_1^2}{\theta^2\kappa^2}(1 - e^{-\kappa\theta})^2 + \frac{\eta_2^2\kappa^2}{\theta^2(\kappa - c)^2}\left(\frac{(1 - e^{-c\theta})}{c} - \frac{(1 - e^{-\kappa\theta})}{\kappa}\right)^2 + \frac{2\rho\eta_1\eta_2}{\theta^2(\kappa - c)}(1 - e^{-\kappa\theta})\left(\frac{(1 - e^{-c\theta})}{c} - \frac{(1 - e^{-\kappa\theta})}{\kappa}\right) \tag{12}$$

As in Balland, the realized variance on Variance Swap strongly depends on mean reversion parameters $\kappa$ and $c$. Thus, catching the correct term structure of Variance Swap variance would provide us accurate values for $\kappa$ and $c$.

## 2.4 Zero-strike Variance Swap variance in Double Heston model

In this section, methods and calculus are exactly the same as those used for above Double log normal Model.

### 2.4.1 Calculation of forward variance in Double Heston: $V_{t,T} = \mathbf{E}_t(V_T)\ \forall\ 0 \le t \le T$

We find the same result as in double log normal:

$$V_{t,T} = \left( V_t - \frac{\kappa(\hat{V}_t - \hat{V}_\infty)}{\kappa - c} - \hat{V}_\infty \right) e^{-\kappa(T-t)} + \frac{\kappa(\hat{V}_t - \hat{V}_\infty)}{\kappa - c} e^{-c(T-t)} + \hat{V}_\infty$$

### 2.4.2 Calculation of the Variance Swap realized variance in Double Heston

We find

$$< \ln(VS(.,.+\theta)), \ln(VS(.,.+\theta)) >_t = \frac{\eta_1^2 (1-e^{-\kappa\theta})^2}{\theta^2\kappa^2} \int_0^t \frac{V_s}{VS(s,s+\theta)^2} ds + \frac{\eta_2^2\kappa^2 \left( \frac{(1-e^{-c\theta})}{c} - \frac{(1-e^{-\kappa\theta})}{\kappa} \right)^2}{\theta^2(\kappa-c)^2} \int_0^t \frac{\hat{V}_s}{VS(s,s+\theta)^2} ds + \frac{2\rho\eta_1\eta_2}{\theta^2(\kappa-c)} (1-e^{-\kappa\theta}) \left( \frac{(1-e^{-c\theta})}{c} - \frac{(1-e^{-\kappa\theta})}{\kappa} \right) \int_0^t \frac{\sqrt{V_s\hat{V}_s}}{VS(s,s+\theta)^2} ds$$

We still make the approximation of an almost flat variance term structure between 0 and t. and we get

$$\begin{aligned} \frac{1}{t} < \ln(VS(.,.+\theta)), \ln(VS(.,.+\theta)) >_t = & \frac{\eta_1^2}{t\theta^2\kappa^2} \left(1 - e^{-\kappa\theta}\right)^2 \int_0^t \frac{1}{VS(s,s+\theta)} ds \\ & + \frac{\eta_2^2\kappa^2}{t\theta^2(\kappa-c)^2} \left( \frac{(1-e^{-c\theta})}{c} - \frac{(1-e^{-\kappa\theta})}{\kappa} \right)^2 \int_0^t \frac{1}{VS(s,s+\theta)} ds \\ & + \frac{2\rho\eta_1\eta_2}{t\theta^2(\kappa-c)} (1-e^{-\kappa\theta}) \left( \frac{(1-e^{-c\theta})}{c} - \frac{(1-e^{-\kappa\theta})}{\kappa} \right) \int_0^t \frac{1}{VS(s,s+\theta)} ds \end{aligned} \tag{13}$$

The term structure of Variance Swap variance is still hardly dependent on the mean reversion parameters.

To use the previous formula we will need to estimate $\int_0^t \frac{1}{VS(s,s+\theta)} ds$

# 3 ATMF Skew asymptotic expansion with respect to volatility of volatility parameters

¶ In this part we use the same method as in [**4**] to derive asymptotic formula for the skew.

First, we start with the dynamics for forward variances fitted to today's forward variance curve. These are specified for a discrete set of periods with tenor $\Delta$,

$$V_t^{i\Delta} = \text{ mean variance between } T_i = t_0 + i\Delta \text{ and } T_{i+1} = t_0 + (i+1)\Delta,$$

as seen from $t$ ($t_0 \leq t \leq T_i$).

Second, we use the link between the smile skew and the skewness of the distribution of logarithmic returns (see [**1**]) to derive the skew. Skewness and kurtosis are defined in terms of cumulants of a distribution of a random variable $X$. Let $\kappa_i$ be the cumulants, i.e. the Taylor coefficients of the generating function. In particular, $\kappa_1 = \mathbb{E}(X)$, $\kappa_2 = Var(X)$, and $\kappa_3 = \mathbb{E}(X - \kappa_1)^3$. The skewness is then defined by

$$\mathcal{S} = \frac{\kappa_3}{(\kappa_2)^{3/2}}.$$

Let us suppose that the normalized logarithmic return of the underlying has a deviation from the normal density quantified by $\mathcal{S}$, then its logarithmic generating function is

$$\frac{s^2}{2!} + \mathcal{S}\frac{s^3}{3!}.$$

and the volatility smile is approximated by $\sigma(K,T) \simeq \sigma_0\big(1 - \frac{\mathcal{S}}{3!}d\big)$ where $d$ is defined by

$$\frac{\log(F/K)}{\sigma_0\sqrt{T}} + \frac{\sigma_0\sqrt{T}}{2}.$$

Thus the ATMF skew $\left.\frac{\partial\sigma}{\partial \log K}\right|_F$ for maturity $T$ is worth

$$\text{Skew}_T = \frac{\mathcal{S}_T}{6\sqrt{T}}.$$

Let try to find an asymptotic expression (when volatility of volatility parameter is small) for the maturity-$T$ skew where $T = N\Delta$. The corresponding logarithmic return is $\log\frac{F_T}{F_0} = \sum_{i=0}^{N-1} r_i$, where the returns $r_i = \log F_{(i+1)\Delta} - \log F_{i\Delta}$. Note that

$$r_i \simeq \sqrt{V_{i\Delta}}(W^F_{(i+1)\Delta} - W^F_{i\Delta}) - \frac{1}{2}V_{i\Delta}\Delta.$$

The second term in this expression is small relative to the first one. Thus we use the following approximations: $r_i^2 = \Delta\, V_{i\Delta}$ and $r_i = \sqrt{V_{i\Delta}}\,(W^F_{(i+1)\Delta} - W^F_{i\Delta})$. The third moment of $\log\frac{F_T}{F_0}$ is

$$\begin{aligned} M_3^T &= \left\langle \left(\log\frac{F_T}{F_0}\right)^3 \right\rangle \\ &= \left\langle \left(\sum_{i=0}^{N-1} r_i\right)^3 \right\rangle \\ &= \left\langle \sum_{i=0}^{N-1} r_i^3 + 3\sum_{i\neq j} r_i r_j^2 + 6\sum_{i\neq j\neq k} r_i r_j r_k \right\rangle \end{aligned}$$

The brackets here stand for expectation.

¶This section is due to Alexandre Engoulatov: alexandre.engoulatov@polytechnique.org

### 3.1 ATMF Skew in Balland's 2 factors

The $j\Delta$-forward instantaneous variance, $V_t^{j\Delta} := \mathrm{E}(V_{j\Delta}/\mathcal{F}_t),\ t \leq j\Delta$ follows (see equation **(7)**)

$$\frac{dV_t^{j\Delta}}{V_t^{j\Delta}} = 2\gamma_{SD}e^{-\lambda_{SD}(j\Delta-t)}dW_t^{SD} + 2\gamma_{LD}e^{-\lambda_{LD}(j\Delta-t)}dW_t^{LD}.$$

In first order with respect to $\gamma_{SD}$ and $\gamma_{LD}$, we deduce that

$$\begin{aligned} V_{j\Delta} &= V_{j\Delta}^{j\Delta} \\ &\simeq V_0^{j\Delta}\left(1 + 2\gamma_{SD}\int_0^{j\Delta} e^{-\lambda_{SD}(j\Delta-u)}dW_u^{SD} + 2\gamma_{LD}\int_0^{j\Delta} e^{-\lambda_{LD}(j\Delta-u)}dW_u^{LD}\right) \end{aligned}$$

And

$$\sqrt{V_{j\Delta}} \simeq \sqrt{V_0^{j\Delta}}\left(1 + \gamma_{SD}\int_0^{j\Delta} e^{-\lambda_{SD}(j\Delta-u)}dW_u^{SD} + \gamma_{LD}\int_0^{j\Delta} e^{-\lambda_{LD}(j\Delta-u)}dW_u^{LD}\right)$$

In order one with respect to $\gamma_{SD}$ and $\gamma_{LD}$, we get

$$\left\langle \sum_{i\neq j\neq k} r_i r_j r_k \right\rangle = 0$$

And

$$\begin{aligned} \left\langle \sum_{i\neq j} r_i r_j^2 \right\rangle &= \sum_{j>i} \Delta \left\langle \sqrt{V_{i\Delta}}V_{j\Delta}\int_{i\Delta}^{(i+1)\Delta} dW_t^F \right\rangle \\ &= \sum_{j>i} \Delta\sqrt{V_0^{i\Delta}}V_0^{j\Delta}\left(2\rho_{SD}\gamma_{SD}\int_{i\Delta}^{(i+1)\Delta} e^{-\lambda_{SD}(j\Delta-u)}du + 2\rho_{LD}\gamma_{LD}\int_{i\Delta}^{(i+1)\Delta} e^{-\lambda_{LD}(j\Delta-u)}du\right) \\ &\simeq \sum_{j>i} 2\Delta^2\sqrt{V_0^{i\Delta}}V_0^{j\Delta}\left(\rho_{SD}\gamma_{SD}e^{-\lambda_{SD}(j-i)\Delta} + \rho_{LD}\gamma_{LD}e^{-\lambda_{LD}(j-i)\Delta}\right) \end{aligned}$$

And finally we write

$$\begin{aligned} \mathcal{S}_\Delta &= \frac{\langle r_i^3 \rangle}{(\langle r_i^2\rangle)^{\frac{3}{2}}} \\ &= \frac{\langle r_i^3 \rangle}{\left(\Delta\, V_0^{i\Delta}\right)^{\frac{3}{2}}} \end{aligned}$$

We thus get

$$\sum_{i=0}^{N-1}\langle r_i^3\rangle = \mathcal{S}_\Delta \sum_{i=0}^{N-1}\left(\Delta\, V_0^{i\Delta}\right)^{\frac{3}{2}}$$

in order one in vol of vol, we easily calculate

$$M_2^T = \left\langle \left(\sum_i r_i\right)^2 \right\rangle = \Delta\sum_i V_0^{i\Delta}$$

The term $\sum_{i=0}^{N-1}\langle r_i^3\rangle$ is of order $\frac{1}{2}$ in $\Delta$, and the term $\left\langle \sum_{i\neq j} r_i r_j^2\right\rangle$ is of order 0. Thus for $\Delta \to 0$, $N\Delta = T$, the first term vanishes and we have

$$\mathrm{Skew}_T = \frac{1}{\sqrt{T}}\Big(\rho_{SD}\gamma_{SD}\zeta(\lambda_{SD}, T) + \rho_{LD}\gamma_{LD}\zeta(\lambda_{LD}, T)\Big) \tag{14}$$

where $\zeta$ is defined by

$$\zeta(\lambda, T) = \lim_{\Delta\to 0,\ N\Delta = T} \left[ \frac{\Delta^2 \sum_{j>i} \sqrt{V_0^{i\Delta}} V_0^{j\Delta} e^{-\lambda(j-i)\Delta}}{\left(\Delta \sum_i V_0^{i\Delta}\right)^{3/2}} \right].$$

Our resulting expression is function of model parameters as well as of today's forward variance curve:

$$\zeta(\lambda, T) = \frac{\int_0^T \int_t^T \sqrt{V_0^t}\, V_0^s e^{-\lambda(s-t)}\, ds\, dt}{\left(\int_0^T V_0^t\, dt\right)^{3/2}}.$$

In particular, if we suppose a flat term structure of variance, we get

$$\text{Skew}_T = \frac{\rho_{SD}\gamma_{SD}}{\lambda_{SD} T} \left(1 - \frac{1}{\lambda_{SD} T}(1 - e^{-\lambda_{SD} T})\right) + \frac{\rho_{LD}\gamma_{LD}}{\lambda_{LD} T} \left(1 - \frac{1}{\lambda_{LD} T}(1 - e^{-\lambda_{LD} T})\right)$$

It's the same formula[‖] found by Gatheral in **[11]**, in the case of a simple Heston model.

For $\lambda T \to 0$,

$$\text{Skew}_T \simeq \frac{\rho_{SD}\gamma_{SD}}{2} + \frac{\rho_{LD}\gamma_{LD}}{2}$$

For $\lambda T \to \infty$,

$$\text{Skew}_T \simeq \frac{\rho_{SD}\gamma_{SD}}{\lambda_{SD} T} + \frac{\rho_{LD}\gamma_{LD}}{\lambda_{LD} T}$$

It follows that if $\lambda T \gg 1$, i.e. the maturity is long relative to the typical mean-reversion time, the skew decays as $1/(\lambda T)$. This is expected because the spot process decorrelates from the volatility process on this time scale. When $\lambda T \sim 0$ (the maturity is short relative to the typical mean-reversion time), the skew is of order one in $\gamma$.

## 3.2 ATMF Skew in double log normal model

Let us derive the corresponding formulas for the double log-normal model. The forward variance follows (One can use equation **(11)** to check that).

$$dV_t^{j\Delta} = \eta_1 V_t e^{-\kappa(j\Delta - t)}\, dW_t^{SD} + \eta_2 \hat{V}_t \frac{\kappa}{\kappa - c} \left(e^{-c(j\Delta - t)} - e^{-\kappa(j\Delta - t)}\right) dW_t^{LD}$$

where $V_t$ is the instantaneous variance and $\hat{V}_t$ is its target variance. We then integrate to get the forward variance in term of instantaneous variance and target variance. We calculate without particular difficulties:

$$\begin{aligned} \left\langle \sqrt{V_{i\Delta}} \int_{i\Delta}^{(i+1)\Delta} dW_t^F\, V_{j\Delta} \right\rangle = \sqrt{V_0^{i\Delta}}\Delta \Big[ & \rho_{SD}\eta_1 e^{-\kappa(j-i)\Delta} G(V_0, \hat{V}_0, \hat{V}_\infty, i\Delta) \\ & + \rho_{LD}\eta_2 \left(e^{-c(j-i)\Delta} - e^{-\kappa(j-i)\Delta}\right) \frac{\kappa}{\kappa - c} G(V_0, \hat{V}_\infty, \hat{V}_\infty, i\Delta) \Big] \end{aligned} \quad (15)$$

where

$$G(z_1, z_2, z_\infty, \tau) = z_\infty + (z_1 - z_\infty)e^{-\kappa\tau} + (z_2 - z_\infty)\frac{\kappa}{\kappa - c}\left(e^{-c\tau} - e^{-\kappa\tau}\right)$$

is the corresponding forward variance functional.

Letting $\Delta \to 0$ we obtain :

$$\text{Skew}_T = \frac{1}{2\sqrt{T}} \Big(\rho_{SD}\eta_1\zeta_1(T) + \rho_{LD}\eta_2\zeta_2(T)\Big)$$

[‖]Without corrections on $\lambda$.

where

$$\zeta_1(T) = \frac{\frac{1}{\kappa}\int_0^T (V_0^t)^{3/2}\left(1-e^{-\kappa(T-t)}\right)\,dt}{\left(\int_0^T V_0^t\,dt\right)^{3/2}}$$

and

$$\zeta_2(T) = \frac{\frac{\kappa}{\kappa-c}\int_0^T \sqrt{V_0^t}\hat{V}_0^t\left(\frac{1-e^{-c(T-t)}}{c} - \frac{1-e^{-\kappa(T-t)}}{\kappa}\right)\,dt}{\left(\int_0^T V_0^t\,dt\right)^{3/2}}$$

As before, $V_0^t$ is today's forward variance curve and $\hat{V}_0^t$ is today's forward target variance curve.

For a flat term structure of variance, we have

$$\text{Skew}_T = \frac{1}{2T}\left\{\left(\frac{\rho_{SD}\eta_1}{\kappa} - \frac{\rho_{LD}\eta_2}{\kappa-c}\right)\left(1-\frac{1}{\kappa T}(1-e^{-\kappa T})\right) + \frac{\rho_{LD}\eta_2\kappa}{c(\kappa-c)}\left(1-\frac{1}{cT}(1-e^{-cT})\right)\right\}$$

For $\lambda T \to 0$, $\quad \text{Skew}_T \simeq \frac{\rho_{SD}\eta_1}{4}$

For $\lambda T \to \infty$, $\quad \text{Skew}_T \simeq \frac{1}{2T}\left(\frac{\rho_{SD}\eta_1}{\kappa} + \frac{\rho_{LD}\eta_2}{\kappa-c}(\frac{\kappa}{c}-1)\right)$

It appears that for very short maturities, the skew depends only on the quantity $\rho_{SD}\eta_1$. Thus we could use this information to drive the short term skew. Actually one of our big challenges is to reduce curvature in the short term maturities forward skew given by our double log normal model.

## 3.3 ATMF Skew in double Heston model

For the Double Heston model, the forward variance follows

$$dV_t^{j\Delta} = \eta_1\sqrt{V_t}e^{-\kappa(j\Delta-t)}\,dW_t^{SD} + \eta_2\sqrt{\hat{V}_t}\frac{\kappa}{\kappa-c}\left(e^{-c(j\Delta-t)} - e^{-\kappa(j\Delta-t)}\right)\,dW_t^{LD}$$

To derive the corresponding Skew formula, we are confronted with evaluating the expectation of $\sqrt{V_t}$ and the expectation of $\sqrt{\hat{V}_t}$. The Dufresne iterative procedure (**[9]**) does not work for this case. However, we can use the formula for the square root (for the proof see for example **[15]** ):

$$\sqrt{\hat{V}} = \frac{1}{2\sqrt{\pi}}\int_0^\infty \frac{1-e^{-s\hat{V}}}{s^{3/2}}ds$$

Passing the expectation under the integral for the positive function $\frac{1-e^{-s\hat{V}}}{s^{3/2}}$ (Fubini), we get

$$\mathbb{E}\left(\sqrt{\hat{V}}\right) = \int_0^\infty \frac{1-\mathbb{E}\left(e^{-s\hat{V}}\right)}{s^{3/2}}ds \tag{16}$$

Now using the fact that $\hat{V}_t$ is affine, we can calculate the Laplace transform of $\hat{V}_t$ density (see **[8]**). We get

$$\mathbb{E}\left(e^{-s\hat{V}_t}\right) = \phi(s)^{\bar{V}}\exp[\lambda_t(\phi(s)-1)],$$

with

$$\bar{V} = \frac{2c\hat{V}_\infty}{\eta_2^2}, \qquad \phi(s) := (1+s\mu_t)^{-1}, \qquad \mu_t = \frac{\eta_2^2}{2}\left(\frac{1-e^{-ct}}{c}\right), \qquad \lambda_t = \frac{2c\hat{V}_0}{\eta_2^2(e^{ct}-1)}.$$

Performing the calculation of **(16)** with expansion with respect to $\eta_2$ small, we get that

$$\mathbb{E}\left(\sqrt{\hat{V}}\right) \simeq \sqrt{\mathbb{E}\hat{V}}$$

Thus, we also approximate $\mathbb{E}\left(\sqrt{V}\right)$ by $\sqrt{\mathbb{E}V}$.

In a similar way that we did for the double log-normal model, we obtain the following expressions for the skew, up to the first order in volatility of volatility:

$$\text{Skew}_T = \frac{1}{2\sqrt{T}}\Big(\rho_{SD}\eta_1\zeta_1(T) + \rho_{LD}\eta_2\zeta_2(T)\Big)$$

where

$$\zeta_1(T) = \frac{\frac{1}{\kappa}\int_0^T V_0^t\left(1-e^{-\kappa(T-t)}\right)dt}{\left(\int_0^T V_0^t\,dt\right)^{3/2}}$$

and

$$\zeta_2(T) = \frac{\frac{\kappa}{\kappa-c}\int_0^T \sqrt{V_0^t\hat{V}_0^t}\left(\frac{1-e^{-c(T-t)}}{c} - \frac{1-e^{-\kappa(T-t)}}{\kappa}\right)dt}{\left(\int_0^T V_0^t\,dt\right)^{3/2}}$$

For a flat term structure of variance, we have

$$\text{Skew}_T = \frac{1}{2T\sqrt{V_0}}\left\{\left(\frac{\rho_{SD}\eta_1}{\kappa} - \frac{\rho_{LD}\eta_2}{\kappa-c}\right)\left(1-\frac{1}{\kappa T}(1-e^{-\kappa T})\right) + \frac{\rho_{LD}\eta_2\kappa}{c(\kappa-c)}\left(1-\frac{1}{cT}(1-e^{-cT})\right)\right\}$$

For $\lambda T \to 0$, $\quad \text{Skew}_T \simeq \frac{\rho_{SD}\eta_1}{4\sqrt{V_0}}$

For $\lambda T \to \infty$, $\quad \text{Skew}_T \simeq \frac{1}{2T\sqrt{V_0}}\left(\frac{\rho_{SD}\eta_1}{\kappa} + \frac{\rho_{LD}\eta_2}{\kappa-c}(\frac{\kappa}{c}-1)\right)$

# 4 Applications and results

In this section, we will use empirical (historical) data to estimate historical Variance Swap variance. Then, for each of our three models, we will use expressions of Variance Swap variance established above to calibrate models parameters.

## 4.1 Historical - Empirical data

For each maturity ($\theta$), We need to estimate the quantity

$$\frac{1}{t} < \ln(VS(.,.+\theta)), \ln(VS(.,.+\theta)) >_t = \lim_{N->\infty}\frac{1}{N\Delta t}\sum_{i=1}^{N}(\ln(\frac{VS(t_i,t_i+\theta)}{VS(t_{i-1},t_{i-1}+\theta)}))^2$$

With $t_0 = 0$ and $t_N = N\Delta t = t$.

Then for each maturity ($\theta$), we need, for every time step, Zero-Strike Variance Swap with maturity ($\theta$). Ideally We should have every day prices on a past period of amount ten years. Unfortunately we are quite a young institution, and we don't have data on a so long past period. We thus need to find some trick to estimate historical Variance Swap (VS).

In **[11]**, Jim Gatheral shows that

$$E(\frac{1}{\theta}\int_0^\theta \sigma_s^2 ds) = \int_{-\infty}^{\infty}\sigma_{BS}^2(z)\mathcal{N}'(z)dz \tag{17}$$

Where $\mathcal{N}$ is the cumulative function of the normal distribution, and $\sigma_{BS}$ is the BS implied volatility for the maturity $\theta$, seen as a function of the log moneyness: $z := \ln(\frac{K}{F})$. And $F$ is the underlying forward for the maturity $\theta$.

Now if we consider the following BS implied variance parametrization :

$$\sigma^2_{BS}(z) = \sigma_0^2 + \alpha z + \beta z^2$$

Using **(17)** we get

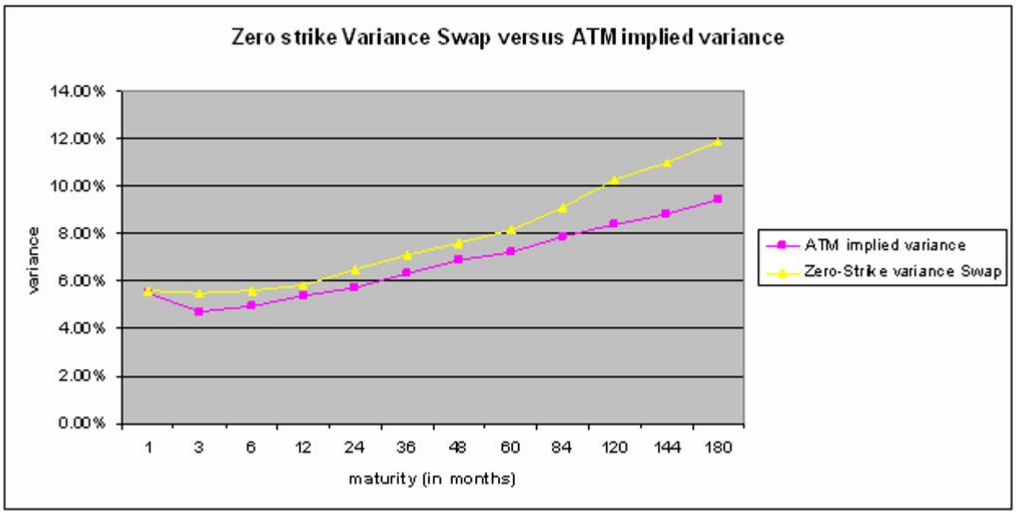
$$E(\frac{1}{\theta}\int_0^\theta \sigma_s^2 ds) = \sigma_0^2 + \beta \tag{18}$$

In the above parametrization of BS implied variance, $\alpha$ is the variance skew and $\beta$ is the variance curve.

We see that the realized variance over a period of time $\theta$ does'nt depend on the skew, but on the curvature. The intuitive explanation of this is that increasing the skew doesn't impact the implied volatility average level, but increasing the curvature increases the convexity of the implied volatility, and increases the volatility out and in the money (on the wings). And that increases the fair level of the volatility.

Finally if we suppose a small curvature for the implied variance[**] ($\beta \simeq 0$), we obtain that a good approximation of realized variance is the at the money forward BS implied variance.

In order to evaluate this approximation, we draw on the next graph the at the money BS implied variance and the Zero-Strike variance Swap for the same maturity. We used a Monte Carlo Pricer[††].

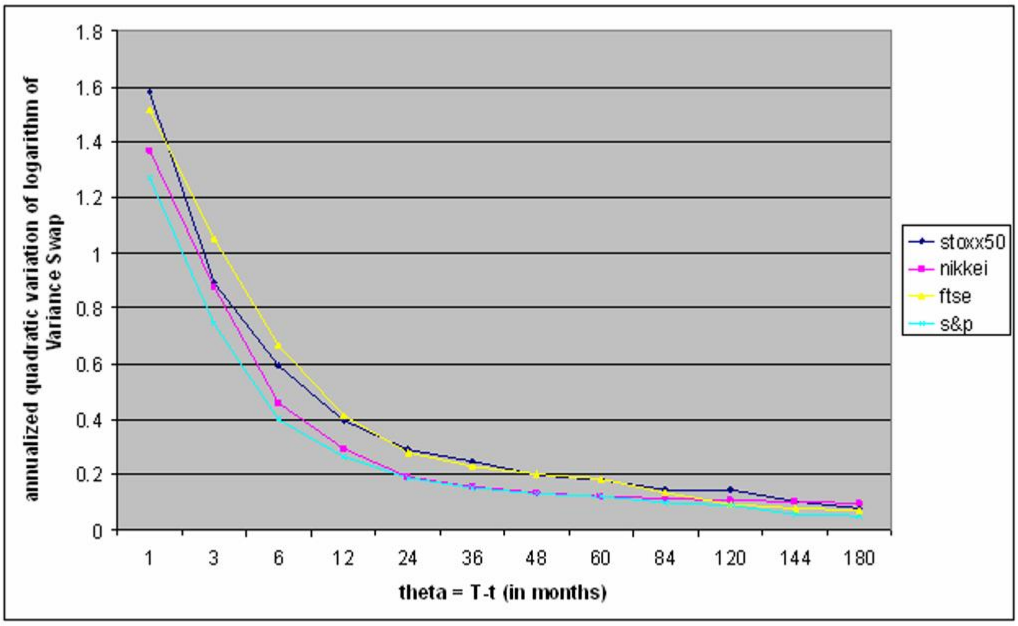



The gap between the two curves above may be the correction term $\beta$ in the the equation **(18)**. In expectation we get $\beta \simeq 1\%$.

[**]This hypothesis is plausible within equities

[††]Double log normal model with 10000 paths, with the set of parameters: $\{\kappa = 740.17\%; c = 10.48\%; \eta_1 = 272\%; \eta_2 = 33.2\%; \rho_S = -87.59\%; \rho_L = -50.62\%; \rho_{SL} = 2.73\%; V_0 = 5.78\%; \hat{V}_0 = 5.20\%; V_\infty = 18.76\%\}$

For Calibration we used Totem[‡‡] monthly data on ATM implied variance, since 1998, to approximate Variance Swap of maturity $\theta$. We got the following term structure of historical Variance Swap variance.




[‡‡]Totem is an institution that gives market consensus from prices provided by different Banks

## 4.2 Calibration of mean reversion parameters on the term structure of implied variance

The calibration then consists in minimizing the quadratic gap between the structure on the previous figure and the analytic formula of Variance Swap variance calculated in the section above. We used a dichotomic algorithm called "Powell algorithm".

One may notice that this method doesn't allow us to fit the whole models parameters. In fact the analytic formula doesn't include the both correlations between the underlying and the volatility brownians. We should find some way to estimate those correlations (by calibrating the asymptotic skew formula for example).

### 4.2.1 Results of calibration in Balland's 2 Factors model

In Balland's 2 factors, we use equation **(8)**. As we can see, $\lambda_{SD}$ and $\lambda_{LD}$ are symmetric in this formula. And also $\gamma_{SD}$ is symmetric to $\gamma_{LD}$. To obtain good calibration, we need to initialize $\lambda_{SD}$ the far as possible from $\lambda_{LD}$. If not we may find same values for $\lambda_{SD}$ and $\lambda_{LD}$, also for $\gamma_{SD}$ and $\gamma_{LD}$. That is not consistent with the model. In particular we won't have time scale separation.
We tried some tricks to separate parameters.

- One idea was to initialize mean reversion parameters one far from other as possible, then while calibrating, we floor $\lambda_{SD}$ and we cap $\lambda_{LD}$. Results are given here

| | **Euro STOXX 50** | **NIKKEI 225** | **S&P 500** | **FTSE 100** |
|---|---|---|---|---|
| $\rho$ | 94.81% | 70.24% | 13.78% | 81.93% |
| $\gamma_{SD}$ | 68.09% | 66.52% | 62.75% | 62.51% |
| $\lambda_{SD}$ | 537.48% | 418.02% | 481.57% | 326.48% |
| $\gamma_{LD}$ | 29.81% | 19.36% | 22.66% | 27.54% |
| $\lambda_{LD}$ | 12.98% | 4.41% | 11.24% | 12.47% |
| $\frac{1}{\lambda_{LD}}$ **(in years)** | 7.70609 | 22.6997 | 8.89291 | 8.01821 |
| $\frac{1}{\lambda_{SD}}$ **(in months)** | 2.23264 | 2.87068 | 2.49184 | 3.67557 |
| $\frac{\lambda_{SD}}{\lambda_{LD}}$ | 41.4186 | 94.8892 | 42.8257 | 26.1778 |
| $\frac{\gamma_{SD}^2}{\lambda_{SD}}$ | 0.0862602 | 0.105862 | 0.0817623 | 0.119701 |
| $\frac{\gamma_{LD}^2}{\lambda_{LD}}$ | 0.68493 | 0.850568 | 0.456486 | 0.607962 |

  We get good results for mean reversion parameters, but correlation between short and long term is globally to high.

- The other idea is that since we want $\lambda_{LD}$ be small, we write a Taylor expansion on **(8)**, with respect to the variable $\lambda_{LD}$, around zero. We get

$$\frac{1}{t} < \ln(VS(.,.+\theta)), \ln(VS(.,.+\theta)) >_t \simeq \frac{1}{\theta^2}\left(\frac{4\gamma_{SD}^2}{\lambda_{SD}^2}(1-e^{-\lambda_{SD}\theta})^2 + 4\gamma_{LD}^2\theta^2(1-\frac{1}{2}\lambda_{LD}\theta)^2\right) + \frac{1}{\theta^2}\left(\frac{8\rho\gamma_{SD}\gamma_{LD}\theta}{\lambda_{SD}}(1-e^{-\lambda_{SD}\theta})(1-\frac{1}{2}\lambda_{LD}\theta)\right) \quad (19)$$

  By doing that, we break the symmetry between the mean reversion parameters and we fit **(19)**. This expansion makes the calibration easier, but we still have huge correlation.
  What we do at end is to fix

$$\lambda_{SD}\lambda_{SD} = \overline{\lambda}$$

  And

$$\gamma_{SD}\gamma_{LD} = \overline{\gamma}$$

At the level of the previous calibration. We replace in **(8)** and we get

$$\frac{1}{t} < \ln(VS(.,.+\theta)), \ln(VS(.,.+\theta)) >_t = \frac{4}{\theta^2}\left( (\frac{\overline{\gamma}}{\overline{\lambda}})^2 (\frac{\lambda_{LD}}{\gamma_{LD}})^2 (1 - e^{-\frac{\overline{\lambda}}{\lambda_{LD}}})^2 + (\frac{\gamma_{LD}}{\lambda_{LD}})^2 (1 - e^{-\lambda_{LD}\theta})^2 \right) + 8\rho \frac{\overline{\gamma}}{\overline{\lambda}\theta^2} (\frac{\lambda_{LD}}{\gamma_{LD}})(1 - e^{-\frac{\overline{\lambda}}{\lambda_{LD}}}) \quad (20)$$

We finally fit **(20)** with only three parameters, rather than five, and we get results in table below

| | **Euro STOXX 50** | **NIKKEI 225** | **S&P 500** | **FTSE 100** |
|---|---|---|---|---|
| $\rho$ | 48.29% | 38.10% | 21.45% | 25.07% |
| $\gamma_{SD}$ | 58.80% | 58.71% | 59.29% | 58.53% |
| $\lambda_{SD}$ | 579.39% | 420.17% | 489.08% | 323.89% |
| $\gamma_{LD}$ | 25.07% | 17.82% | 19.88% | 23.60% |
| $\lambda_{LD}$ | 7.33% | 2.46% | 6.97% | 7.70% |
| $\frac{1}{\lambda_{LD}}$ **(in years)** | 13.64671653 | 40.70186292 | 14.3414 | 12.9828 |
| $\frac{1}{\lambda_{SD}}$ **(in months)** | 2.07112949 | 2.856014052 | 2.45359 | 3.70495 |
| $\frac{\lambda_{SD}}{\lambda_{LD}}$ | 79.06825678 | 171.0153894 | 70.1407 | 42.05 |
| $\frac{\gamma_{SD}^2}{\lambda_{SD}}$ | 0.059669932 | 0.082027665 | 0.0718658 | 0.10576 |
| $\frac{\gamma_{LD}^2}{\lambda_{LD}}$ | 0.857463351 | 1.293034209 | 0.566688 | 0.722834 |

Results are satisfactory. Mean reversion parameters are consistent with time scale. To see how good is the fit, in appendix A, we represent on the same graph historical curve and model curve for each of our four indexes. We observe that Balland 2 factors (as the others 2 factors models studied in this report) fit very well the historical term structure of Variance variance.

### 4.2.2 Results of calibration in Double log normal

We use here the equation **(12)**. There is no particular difficulty and we get the following results

| | **Euro STOXX 50** | **NIKKEI 225** | **S&P 500** | **FTSE 100** |
|---|---|---|---|---|
| $\rho$ | 86.48% | 29.98% | 55.37% | 24.83% |
| $\kappa$ | 740.17% | 470.92% | 416.93% | 296.75% |
| $\eta_1$ | 152.71% | 133.35% | 135.10% | 137.20% |
| c | 10.48% | 11.61% | 3.13% | 14.08% |
| $\eta_2$ | 52.82% | 44.96% | 36.40% | 56.90% |
| $\frac{1}{\kappa}$ **(in months)** | 1.621249172 | 2.548230572 | 2.878194796 | 4.043835173 |
| $\frac{1}{c}$ **(in years)** | 9.541347429 | 8.609778987 | 31.97482941 | 7.103836782 |
| $\frac{\eta_1^2}{\kappa}$ | 0.3150509 | 0.377593035 | 0.437799592 | 0.634356046 |
| $\frac{\eta_2^2}{c}$ | 2.662071152 | 1.740598882 | 4.235908524 | 2.300284848 |

Results are good enough.

#### 4.2.3 Results of calibration in Heston 2 Factors

We use equation **(13)**, and we numerically estimate the quantity $\int_0^t \frac{1}{VS(s,s+\theta)}ds$ as:

$$\int_0^t \frac{1}{VS(s,s+\theta)}ds \simeq \Delta t \sum_{i=1}^{N} \frac{1}{VS(t_i,t_i+\theta)}$$

Results of calibration are the follows:

| | **Euro STOXX 50** | **NIKKEI 225** | **S&P 500** | **FTSE 100** |
|---|---|---|---|---|
| $\rho$ | 95.94% | 98.14% | 93.56% | 96.39% |
| $\kappa$ | 1169.89% | 507.18% | 429.38% | 542.22% |
| $\eta_1$ | 24.30% | 16.12% | 24.25% | 17.09% |
| c | 9.56% | 4.83% | 1.91% | 9.64% |
| $\eta_2$ | 12.11% | 8.10% | 7.57% | 10.69% |
| $\frac{1}{c}$ **(in years)** | 10.4602 | 20.7 | 52.2234 | 10.3759 |
| $\frac{1}{\kappa}$ **(in months)** | 1.02574 | 2.36602 | 2.79475 | 2.21312 |
| $\frac{\kappa}{c}$ | 122.372 | 104.986 | 224.235 | 56.2601 |
| $\frac{\eta_1^2}{\kappa}$ | 0.005049 | 0.00512105 | 0.0136948 | 0.00538753 |
| $\frac{\eta_2^2}{c}$ | 0.153314 | 0.135806 | 0.299649 | 0.118622 |

# Conclusion

In this report, we worked on 3 particular 2 factors stochastic volatility models: Balland's model, Gatheral's model (double Log Normal) and double Heston model. We provided formula for forward Variance Swap variance. We also provided an asymptotic expansion of the ATMF skew, with respect to the volatility of volatility parameters around zero.

We estimated empirical Variance Swap by the ATMF implied variance for the same maturity. We used totem data to calculate term structure of historical Variance Swap variance. Then we used closed form formula on Variance Swap variance to fit the historical term structure of implied variance. That allowed us to calibrate some parameters of the models. We got satisfactory results, especially for the mean reversion parameters.

In the future, the next step of our work could be to use asymptotic skew formula calculated here to fit the product of the volatility of volatility parameter with the corresponding Underlying-correlation parameter. We could also calculate asymptotic formula for vanilla prices, as it's done in [**10**].

# References


[1] D. Backus, S. Foresi, and L. Wu. *Accounting for Biases in Black-Scholes.* August 31, 2004.

[2] Philippe Balland. *Forward Smile.* Global Derivatives 2006. Paris, 2006.

[3] Lorenzo Bergomi.*Smile Dynamics.* Risk. September 2004.

[4] Lorenzo Bergomi.*Smile Dynamics II.* Risk. October 2005.

[5] S. Charroz, Et A. Daerr. *Algorithmes de minimisation.* Université Paris 7 Denis Diderot-CEA Saclay.

[6] Claude Delannoy. *Apprendre le C++.* Eyrolles. 2007.

[7] K. Demeterfi, E. Derman, M. Kamal, J. Zou. *More Than You Ever Wanted To Know About Volatility Swaps.* Goldman Sachs, Quantitative Strategies Research Notes. March 1999.

[8] Darrell Duffie. *Financial Modeling With Affine Processes, Preliminary and Incomplete Draft.* Standford University and University of Lausanne. January 15, 2008.

[9] Daniel Dufresne. *The Integrated Square-Root Process.* University of Montreal. Research paper Number 90. November 2001.

[10] J.-P. Fouque, G. Papanicolaou, K.R. Sircar, and K. Solna. *Multiscale Stochastic Volatility Asymptotics.*

[11] Jim Gatheral.*Case Studies in Financial Modeling Course Notes.* Courant Institute of Mathematical Sciences. Fall Term, 2005.

[12] Jim Gatheral. *Developments in Volatility Derivatives Pricing.* Global Derivatives 2007. Paris, May 23, 2007.

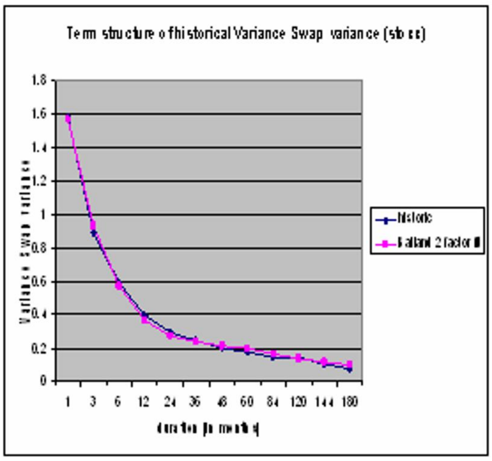
[13] Jim Gatheral. *Further Developments in Volatility Derivatives Modeling.* Global Derivatives Trading and Risk Management. Paris,May 21, 2008.

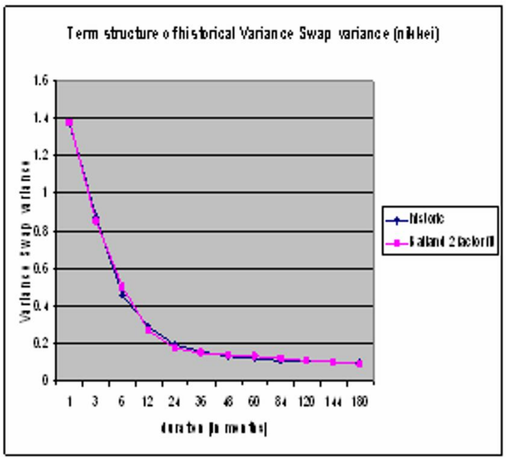
[14] P.J. Salzman. *Gaussian Quadrature.* June 1, 2001.

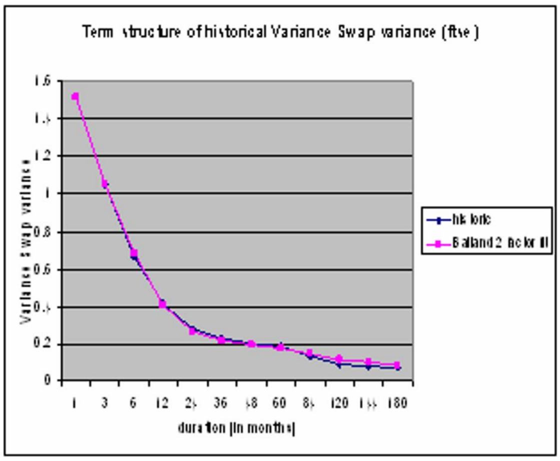
[15] Klaus Schürger. *Laplace Transfors and Suprema of Stochastic Processes.* University of Bonn. 2002.

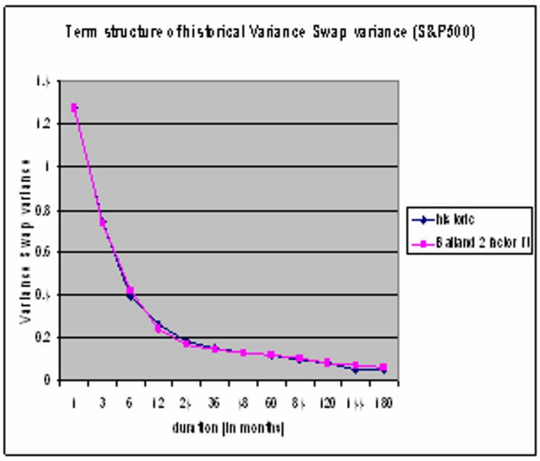
# A Historical Variance Swap variance versus Models Variance Swap variance

## A.1 Balland 2 Factors

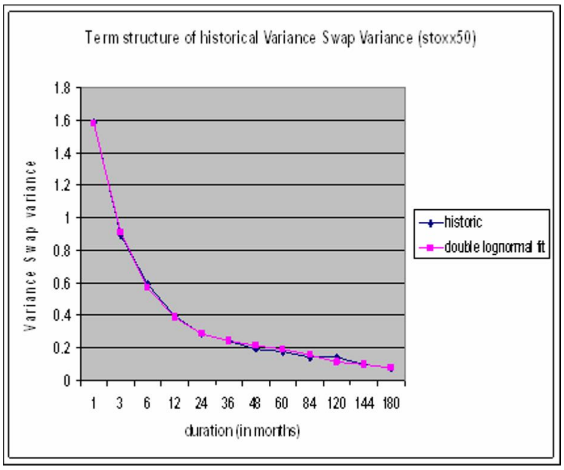


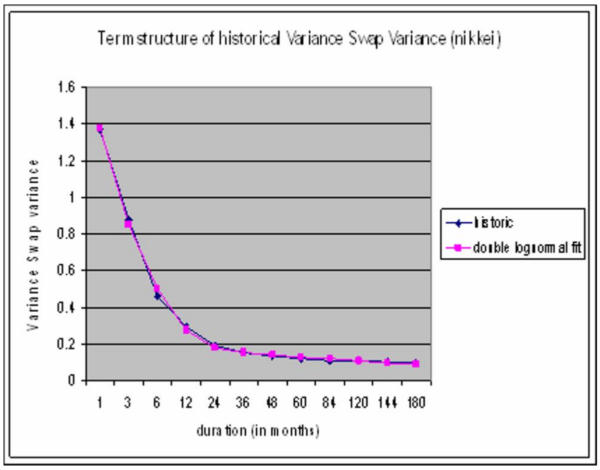



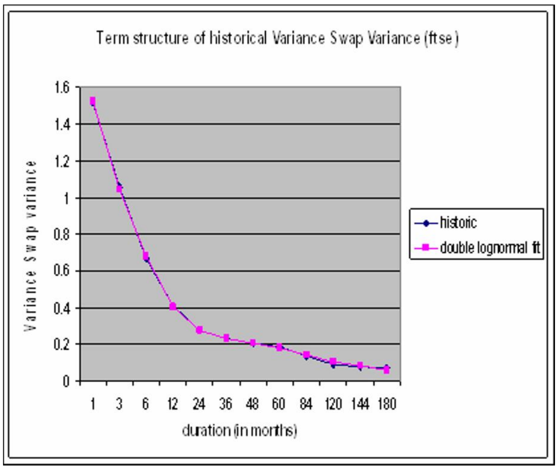



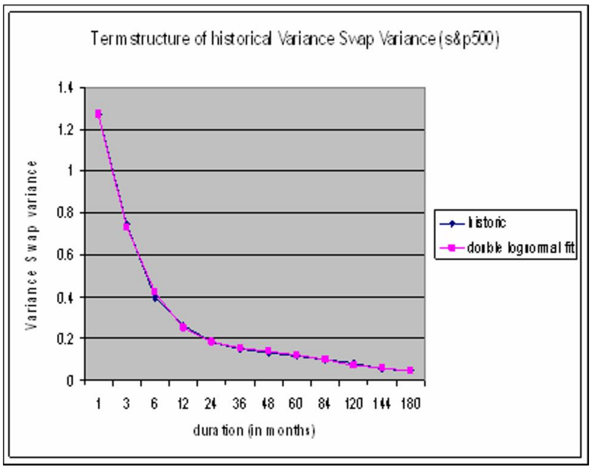




## A.2 Double Log Normal

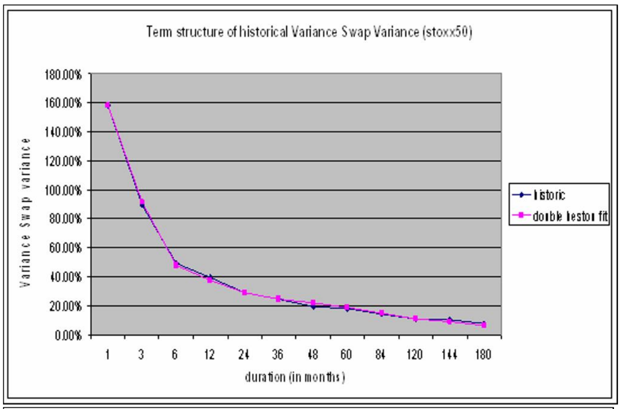


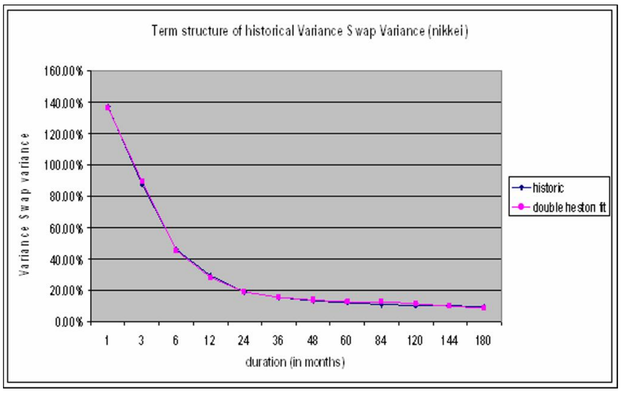



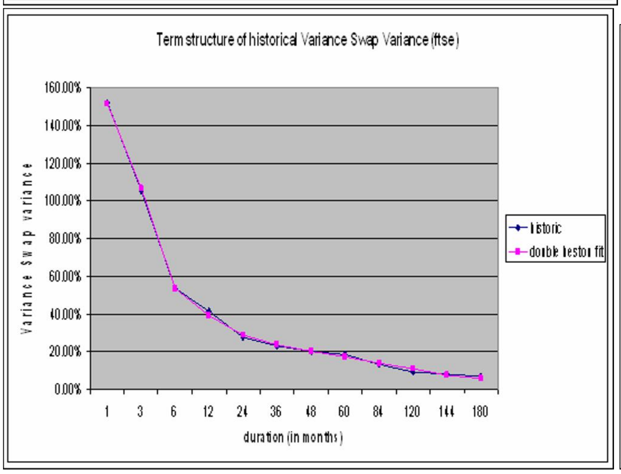


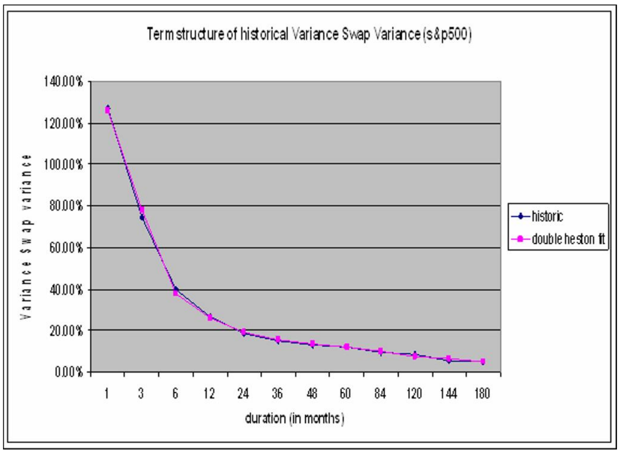




## A.3 Double Heston



# B ATMF asymptotic skew versus Monte Carlo skew in Double Log Normal Model