Calibration on historical term structure of Spot-Variance correlation*

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8th October 2009

*This note is entirely inspired by [1]

1 Preliminaries

Let consider the following process:

$$\begin{aligned}\frac{dS_t}{S_t} &= rdt + \sigma_t dW_t^S \\ \frac{dV_t}{V_t} &= \alpha_t dt + \eta_t dW_t^V \\ d < W^S, W^V >_t &= \rho dt\end{aligned}$$

Then, the realized correlation between S and V yields is

$$\begin{aligned}\rho_t(S,V) &= \frac{Cov(\int_0^t \frac{dS_u}{S_u}, \int_0^t \frac{dV_h}{V_h})}{\sqrt{Var(\int_0^t \frac{dS_u}{S_u})Var(\int_0^t \frac{dV_u}{V_u})}} \\ &= \frac{\mathbb{E} < \ln S, \ln V >_t}{\sqrt{\mathbb{E} < \ln S, \ln S >_t \mathbb{E} < \ln V, \ln V >_t}} \\ &= \frac{\rho \int_0^t \mathbb{E}(\sigma_u \eta_u) du}{\sqrt{\int_0^t \mathbb{E}\sigma_u^2 du \int_0^t \mathbb{E}\eta_u^2 du}}\end{aligned}$$

Cauchy-Schwartz inequality gives

$$\int_0^t \mathbb{E}(\sigma_u \eta_u) du \leq \sqrt{\int_0^t \mathbb{E}\sigma_u^2 du \int_0^t \mathbb{E}\eta_u^2 du}$$

And then.

$$|\rho_t(S,V)| \leq |\rho|$$

We find that the absolute value of the average realized correlation is always smaller than the absolute value of the correlation seen as a model parameter. In particular the historical realized correlation is always smaller than the absolute value of the correlation seen as the model parameter.

2 Calibration of double lognormal parameters on historical term structure of Spot-Variance Swap correlation

We use variance Swap rather than underlying's variance because Variance Swap is liquid, whereas underlying's variance is not measurable.

2.1 Historical term structure of Spot-Variance Swap correlation

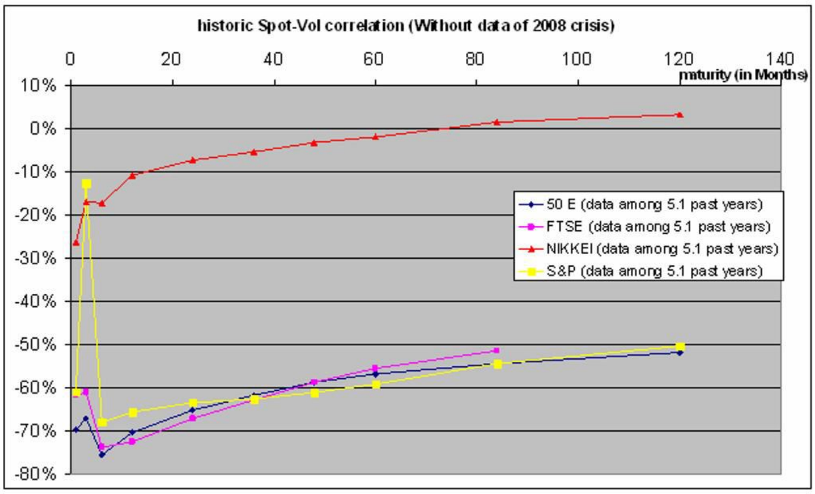
To calculate the historical correlation between the spot and a Variance Swap for a given maturity θ, within a period of time $[0, t]$, we use the results in section **1**:

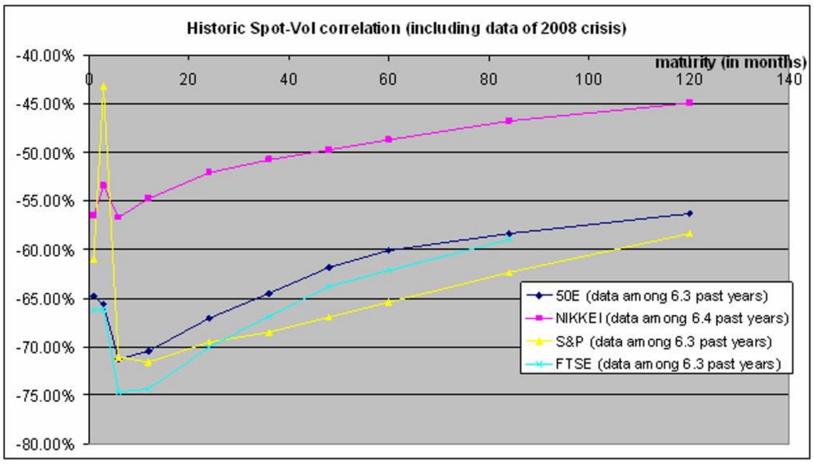
$$\rho_t(S, VS(\theta)) = \frac{\sum_{i=1}^N \ln\left(\frac{VS(t_i, t_i+\theta)}{VS(t_{i-1}, t_{i-1}+\theta)}\right) \ln\left(\frac{S_{t_i}}{S_{t_{i-1}}}\right)}{\sqrt{\sum_{i=1}^N \ln\left(\frac{VS(t_i, t_i+\theta)}{VS(t_{i-1}, t_{i-1}+\theta)}\right)^2 \sum_{i=1}^N \ln\left(\frac{S_{t_i}}{S_{t_{i-1}}}\right)^2}}$$

With

$$t = \sum_{i=1}^N (t_i - t_{i-1})$$

We use monthly data of ATMSpot BS variance, and we get the following term structure of correlation





2.2 Formula for the "Spot-Variance Swap" correlation in double log normal model

Let remind that in Gatheral model, variance Swap diffuses as follow:

$$dVS(t,t+\theta) = \phi_{t,\theta}dt + \frac{\eta_1 V_t}{\kappa\theta}(1-e^{-\kappa\theta})dW_t^S + \frac{\kappa\eta_2\hat{V}_t}{\theta(\kappa-c)}\left(\frac{(1-e^{-c\theta})}{c} - \frac{(1-e^{-\kappa\theta})}{\kappa}\right)dW_t^L$$

With

$$\phi_{t,\theta} = \frac{(1-e^{-\kappa\theta})}{\theta}\left(\hat{V}_t - V_t - \frac{c}{\kappa-c}(V_\infty - \hat{V}_t)\right) + \frac{\kappa(1-e^{-c\theta})}{\theta(\kappa-c)}(V_\infty - \hat{V}_t)$$

The Spot diffusion is given by the equation

$$\frac{dS_t}{S_t} = rdt + \sqrt{V_t}dW_t^F$$

Then we get

$$d < \ln VS(.,.+\theta), \ln S >_t = \frac{\rho_S\eta_1\sqrt{V_t}V_t}{\kappa\theta VS(t,t+\theta)}(1-e^{-\kappa\theta})dt + \frac{\kappa\rho_L\eta_2\sqrt{V_t}V_t}{\theta(\kappa-c)VS(t,t+\theta)}\left(\frac{(1-e^{-c\theta})}{c} - \frac{(1-e^{-\kappa\theta})}{\kappa}\right)dt$$

We make the approximation that in expectation, the level of V is close the the Variance Swap. This leads to

$$\rho_t(S,VS(\theta)) = \frac{\mathbb{E}\langle \ln S, \ln VS(.,.+\theta)\rangle_t}{\sqrt{\mathbb{E}\langle \ln S, \ln S\rangle_t \mathbb{E}\langle \ln VS(.,.+\theta), \ln V(.,.+\theta)\rangle_t}}$$

$$= \frac{\frac{\rho_S\eta_1}{\kappa\theta}(1-e^{-\kappa\theta})+\frac{\kappa\rho_L\eta_2}{\theta(\kappa-c)}\left(\frac{(1-e^{-c\theta})}{c}-\frac{(1-e^{-\kappa\theta})}{\kappa}\right)}{\sqrt{\frac{\eta_1^2}{\kappa^2\theta^2}(1-e^{-\kappa\theta})^2+\frac{\kappa^2\eta_2^2}{\theta^2(\kappa-c)^2}\left(\frac{(1-e^{-c\theta})}{c}-\frac{(1-e^{-\kappa\theta})}{\kappa}\right)^2+\frac{2\rho\eta_1\eta_2}{\theta^2(\kappa-c)}(1-e^{-\kappa\theta})\left(\frac{(1-e^{-c\theta})}{c}-\frac{(1-e^{-\kappa\theta})}{\kappa}\right)}} \frac{\mathbb{E}\int_0^t \sqrt{V_u}du}{\sqrt{t\mathbb{E}\int_0^t V_u du}}$$

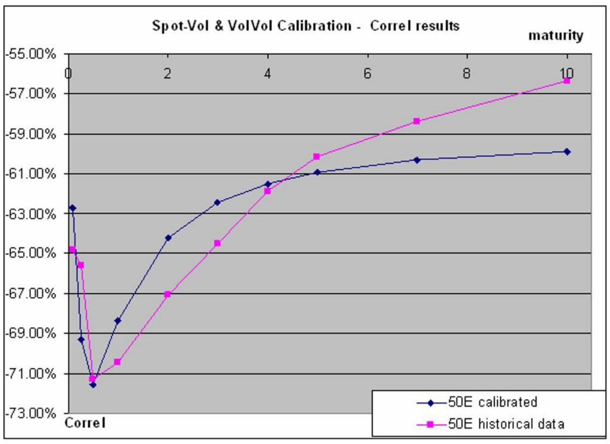
And if we neglect the second factor, we finally get

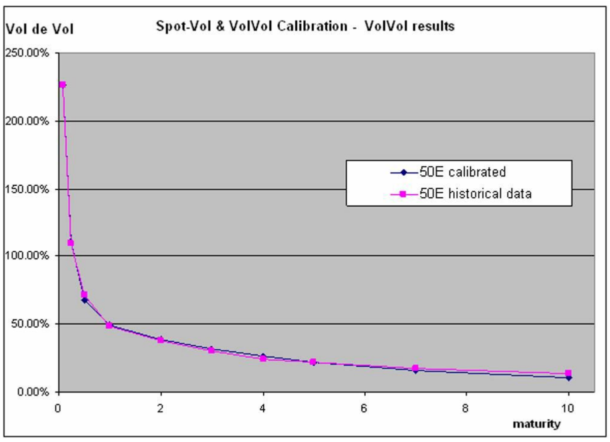
$$\rho_t(S,VS(\theta)) = \frac{\frac{\rho_S\eta_1}{\kappa\theta}(1-e^{-\kappa\theta})+\frac{\kappa\rho_L\eta_2}{\theta(\kappa-c)}\left(\frac{(1-e^{-c\theta})}{c}-\frac{(1-e^{-\kappa\theta})}{\kappa}\right)}{\sqrt{\frac{\eta_1^2}{\kappa^2\theta^2}(1-e^{-\kappa\theta})^2+\frac{\kappa^2\eta_2^2}{\theta^2(\kappa-c)^2}\left(\frac{(1-e^{-c\theta})}{c}-\frac{(1-e^{-\kappa\theta})}{\kappa}\right)^2+\frac{2\rho\eta_1\eta_2}{\theta^2(\kappa-c)}(1-e^{-\kappa\theta})\left(\frac{(1-e^{-c\theta})}{c}-\frac{(1-e^{-\kappa\theta})}{\kappa}\right)}} \quad (1)$$

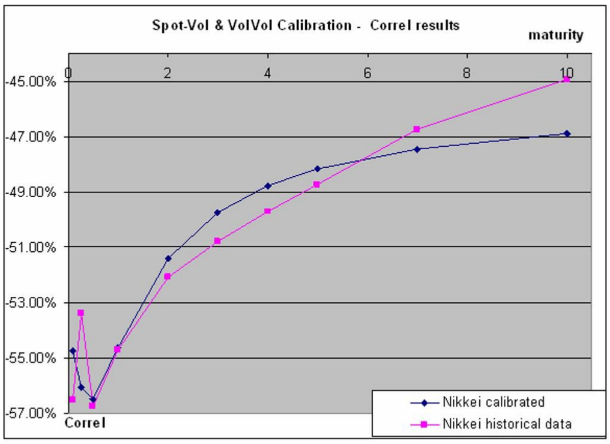
2.3 Calibration

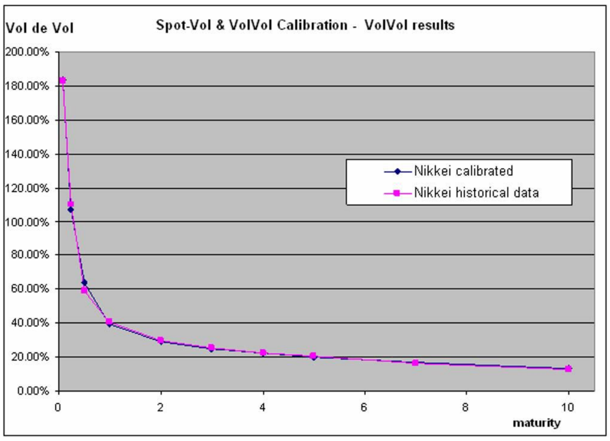
The calibration consist in finding the parameters such that the analytic formula in equation **(1)** fits the historical term structure in section **2.1**. For that, we minimize the quadratic gap between the analytic formula and the historical data. We calibrate only the three parameters ρ_S, ρ_L and ρ_{SL} on the historical term structure of correlation. The other parameters are calibrated on Volatility of Volatility term structure. We thus get the historical calibration set of parameters below.

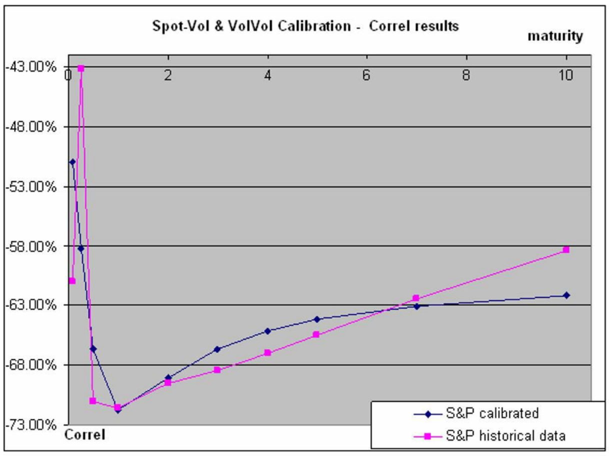
	STOXX50	**NIKKEI225**	**S&P500**	**FTSE100**
κ	763.26%	538.19%	454.91%	334.52%
c	20.35%	8.83%	15.64%	14.21%
η_S	194.39%	160.74%	160.99%	159.22%
η_L	73.89%	52.58%	57.07%	63.37%
ρ_S	-58.00%	-53.76%	-47.26%	-62.42%
ρ_L	-56.73%	-44.62%	-57.38%	-51.99%
ρ_{SL}	28.43%	56.14%	7.33%	16.70%

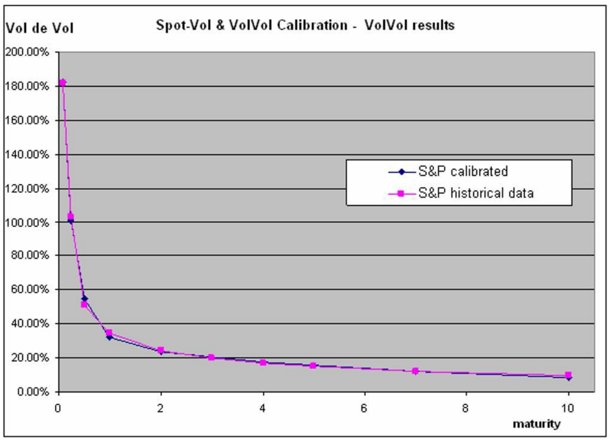


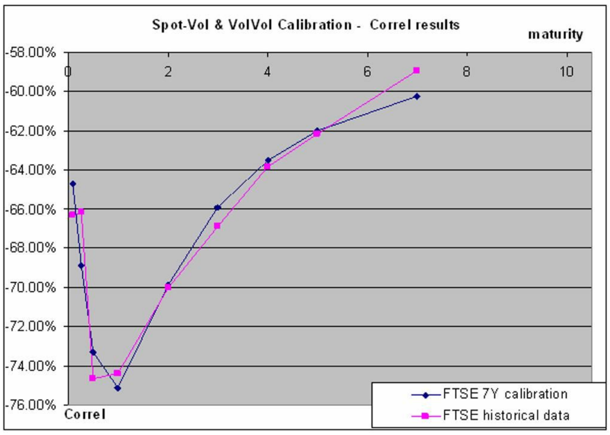


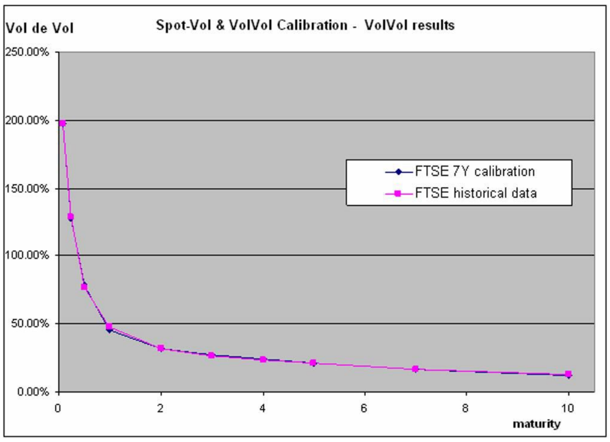












References

[1] KOUOKAP YOUMBI D.: October 2008, 2 Factors Stochastic Volatility Models: Formula for the Variance Swap variance and asymptotic formula for the At The Money Forward (ATMF) Skew, *Internship Report*