Average Yield to Maturity Curve Construction

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Abstract

This note explains how to interpolate and extrapolate, with respect to maturity date, the yield to maturity curve (term structure), from market quotes of yield points, for different maturities

1 Introduction

For a given issuer, market quotes yield to maturities relative to bonds issued. However these quotes do not clearly follow a smooth line in terms of maturities. It looks like they are quoting above (cheap) or below (rich) an average smoothed and time continuous curve (see figure 1 below).

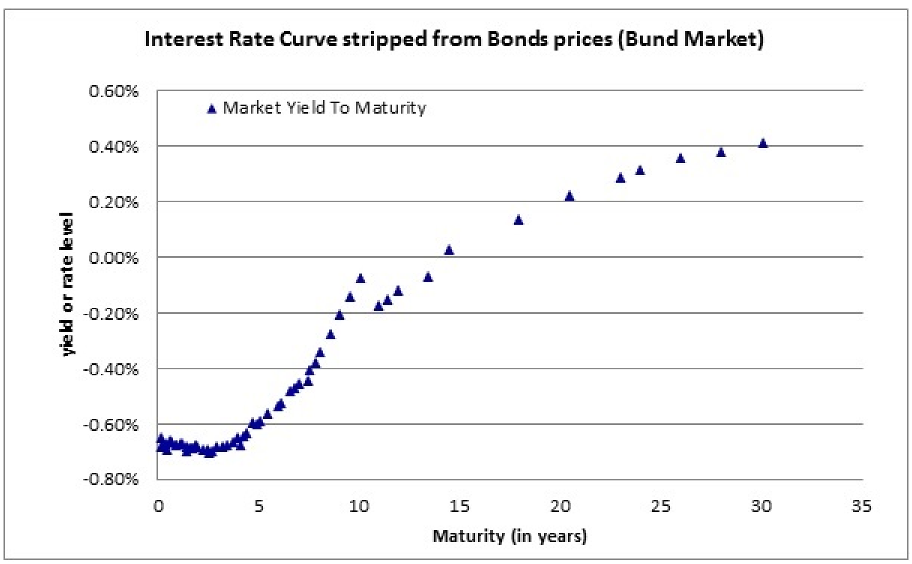


Figure 1: Market Bund Curve (term structure), as of 13/07/2016. Source: Bloomberg

The interest of extracting the 'accurate' average yield curve is multiple:

- It could help a trader assessing on live, which bonds are trading cheap and which ones are trading expensive; and therefore buy or sell accordingly;
- It could help interpolating/extrapolating bond prices for maturities where there is no market quote available;
- It could help calculating a measure of liquidity within the given market;
- It could help calculating some Bonds' sensitivities and use this for dynamically hedging a position;
- It sysmatically gives us a forward short rate curve, that could be observed to predict the most-likely path of the short rate in the future.

Our approach will consist in three steps: In the first step we will choose a model (Vasicek 2 Factors) for the short interst rate. In the second step we will compute the theoretical bond price in this model, and finally in the third step we will calibrate the model parameters to minimise the total squares of the spreads between the model theoretical yields-to-maturity and the market quoted yields-to-maturity.

2 The two factors Vasicek model for the short interest rate

Let r be the value of the short interest rate. Then we will postulate the following diffusion equations for the short rate:

1

$$\begin{aligned} dr_t &= \kappa\left(\bar{r}_t - r_t\right)dt + \sigma_r dW_t^r \\ d\bar{r}_t &= \lambda\left(r_\infty - \bar{r}_t\right)dt + \sigma_{\bar{r}} dW_t^{\bar{r}} \\ d \prec W^r, W^{\bar{r}} \succ_t &= \rho(t)dt \end{aligned} \tag{1}$$

Where:

- $\bar{r}$ is the level towards which the short interest rate reverts regularly. This level is it self stochastic, and change with time, following the second equation in 1
- $(W^r)_{t\geq 0}$ and $(W^{\bar{r}})_{t\geq 0}$are two brownian motions driving the uncertainties on the short rate value and its mid-term level respectively. ρ is the instantaneous correlation between these two brownian motions. ρ can also be see as the correlation between the short-term interest rate and the mid-term interest rate ;
- κ; σ_r; ;λ; r_∞; $\sigma_r and$ $\sigma_{\bar{r}}$ are the model's parameters, and should be calibrated to fit the market yield term structure displayed on figure 1 above;

- κ is the mean reversion parameter. It drives how quick the short rate process reverts towards its mid-term level $\bar{r}$. The bigger is κ, the quicker the short rate process reverts towards $\bar{r}$;
- $\bar{r}$ periodically reverts towards $r_\infty$with intensity λ;
- r_∞ is the long term level towards which the short rate process will converge (in expectation) after a period of diffusion;
- σ_r and $\sigma_{\bar{r}}$ are two parameters driving the instantaneous volatility of r and $\bar{r}$ respectively;

Dynamics in equation 1 is referred to as double Vasicek model. In this model the short rate has a normal distribution, and is therefore allowed to take negative values. This is really important as we now have rates quoting negative, in the market (see the short end of the curve in figure 1 above).

3 Bond Price in two factors Vasicek model

In this section We will derive the expression of a bond price in the two factors Vasicek model. Let $P(t,T)$ be the price at time t of a zero-coupon bond of maturity T: the value of a bond paying no coupon, and paying 1 at maturity date. By definition we have that

$$P(t,T) := \mathbb{E}_t\left[e^{-\int_t^T r_s ds}\right]$$

Let $B(t,T)$ be the price at time t of a bond paying an annual coupon c, η times per year. Then the fair-value of this bond is (approximatively):

$$\begin{aligned} B(t,T) &= \sum_{T_i=T_1}^{T} \frac{c}{\eta} P(t,T_i) + P(t,T) \\ &= \frac{c}{\eta} \sum_{T_i=T_1}^{T} P(t,T_i) + P(t,T) \end{aligned}$$

A quick calculation gives that the value of a zero-coupon bond is

$$P(t,T) = e^{\left\{-\tau(t,T,\kappa)r_t - C(t,T)\bar{r}_t - D(t,T)r_\infty + \frac{1}{2}Var\left(\tilde{\mathcal{N}}(t,T)\right)\right\}} \tag{2}$$

With

$$\begin{aligned} \tau(t,T,\kappa) &= \frac{1}{\kappa}\left(1 - e^{-\kappa(T-t)}\right) \\ C(t,T) &= \frac{\kappa}{\kappa-\lambda}\left(\tau(t,T,\lambda) - \tau(t,T,\kappa)\right) \\ D(t,T) &= T - t - C(t,T) - \tau(t,T,\kappa) \end{aligned}$$

And

$$Var\left(\tilde{\mathcal{N}}(t,T)\right) = \left[\left(\frac{\sigma_r}{\kappa}\right)^2 + \left(\frac{\sigma_{\bar{r}}}{\kappa(\kappa-\lambda)}\right)^2 - 2\rho\frac{\sigma_r\sigma_{\bar{r}}}{\kappa^2(\kappa-\lambda)}\right] I(t,T,\kappa) + \left(\frac{\sigma_{\bar{r}}}{\lambda(\kappa-\lambda)}\right)^2 I(t,T,\lambda) - 2\frac{\sigma_{\bar{r}}}{\lambda(\kappa-\lambda)}\left(\frac{\sigma_{\bar{r}}}{\kappa(\kappa-\lambda)} - \rho\frac{\sigma_r}{\kappa}\right) J(t,T,\kappa,\lambda)$$

$$I(t,T,\kappa) = T - t - 2\tau(t,T,\kappa) + \tau(t,T,2\kappa)$$

$$J(t,T,\kappa,\lambda) = T - t - \tau(t,T,\kappa) - \tau(t,T,\lambda) + \tau(t,T,\kappa+\lambda)$$

4 From Bond prices to Yield-To-Maturity

Now we have Bonds' prices expressed as a function of the model parameters, we can compute the Yield-To-Maturity. Let $y(t,T)$ denote the Yield-To-Maturity associated with the bond price $B(t,T)$. Then the relationship between the Bond price and the Yield-To-Maturity is the following (assuming the interest rate is compounded):

$$B(t,T) = \frac{1}{(1+y)^{(T_1(t)-t)}}\left[c\left(T_1(t)-t\right) + \frac{c}{\eta}\frac{1-(1+y)^{-(T-T_1(t))}}{(1+y)^{\frac{1}{\eta}}-1}\right] + \frac{1}{(1+y)^{(T-t)}} \tag{3}$$

Where $T_1(t)$ is the first coupon date from todat (t).

In particular for the zero-coupon Bond we have that

$$P(t,T) := \frac{1}{(1+y)^{(T-t)}}$$

Given the price, the Yield-To-Maturity is calculated by inverting equation 3. One could use optimised Dichotomy or Newton-Raphson algorithms. In the case of zero-coupon Bond we have that

$$\begin{aligned} y(t,T) &= e^{-\frac{1}{T-t}\ln(P(t,T))} - 1 \\ &= \frac{1}{P(t,T)^{\frac{1}{T-t}}} - 1 \end{aligned} \tag{4}$$

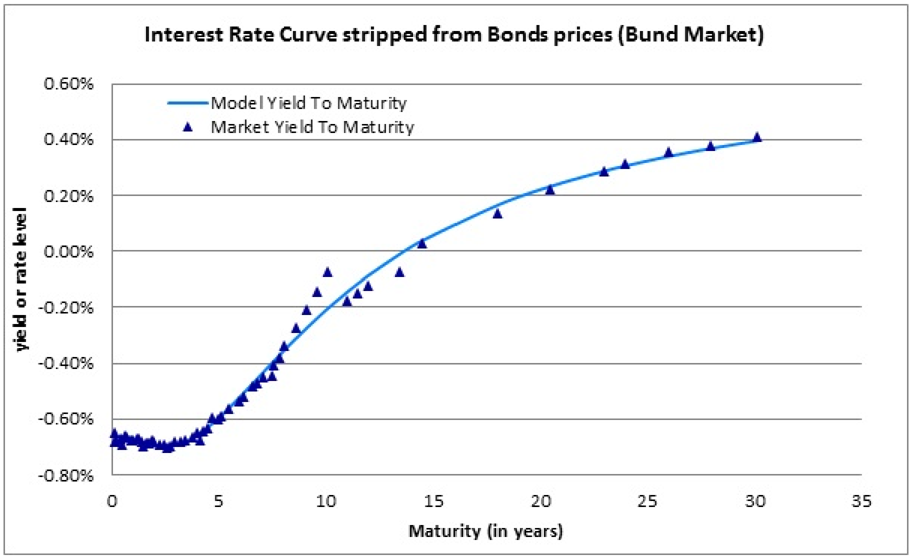
5 Calibration

The calibration will consist in finding the model parameters' values such that the sum of the squares of the spread between the model's Yield-To-Maturity and the market Yield-To-Maturity is minimal.

Bonds quoted in the market are not necessary zero-coupon bonds. For the sake of simplicity and for reducing the computation time, we might want to use zero-coupon bonds only, in our calculations. This is possible because from non-arbitrage arguments, one can assume that it is always possible to exhibit (i.e determine the price) a zero-coupon bond having the same yield as a given non-zero-coupon bond. We therefore use equations 2 and 4 to compute the model Yield-To-Maturity that will fit market points given in Figure 1 above. Results are given in Figure 2 below:

Vasicek - 2 Factors	Calibration
Parameter	Value
Initial Short Rate	-0.68%
Short Rate Mid Term Value	-0.30%
Short Rate Long Term Value	1.45%
Rate Short Term Mean Return	41.44%
Rate Long Term Mean Return	22.63%
Rate Short Term Volatility	8.88%
Rate Long Term Volatility	2.09%
Short Term - Long Term Correlation	-85.35%

(a) Calibrated Parameters



(b) Fit Quality

Figure 2: Market Yield-To-Maturity term structure calibrated

6 Forward Short Rate

From the zero-coupon bond expression in quation 2, we can derive a forward short rate term structure as follow:

$$f(t,T) = -\frac{\partial \ln P}{\partial T}(t,T)$$

After calculation we get that

$$f(t,T) = r_t - \kappa\tau(t,T,\kappa)\left(r_t - r_\infty\right) + \frac{\kappa}{\kappa-\lambda}\left(\kappa\tau(t,T,\kappa) - \lambda\tau(t,T,\lambda)\right)\left(\bar{r}_t - r_\infty\right) - \frac{1}{2}V(t,T)$$

With

$$V(t,T) := \left(\sigma_r^2 + \left(\frac{\sigma_{\bar{r}}}{\kappa-\lambda}\right)^2 - 2\rho\frac{\sigma_r\sigma_{\bar{r}}}{\kappa-\lambda}\right)\tau(t,T,\kappa)^2 + \left(\frac{\sigma_{\bar{r}}}{\kappa-\lambda}\right)^2\tau(t,T,\lambda)^2 - 2\frac{\sigma_{\bar{r}}}{\kappa-\lambda}\left(\frac{\sigma_{\bar{r}}}{\kappa-\lambda} - \rho\sigma_r\right)\tau(t,T,\kappa)\tau(t,T,\lambda)$$

Computation Results are given in Figure 3 below:

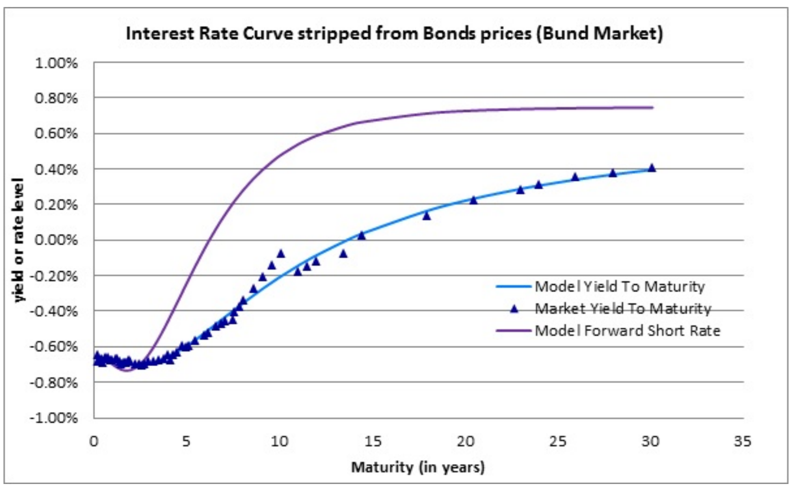


Figure 3: Model Fit and Forward Short Rate

7 Conclusion

In this note we have explained how to fit a bond market Yield-To-Maturity curve, assuming a two factors Vasicek model for the short interest rate dynamics. We have derived a closed form formula for the zero-coupon bond, and for the forward short interest rate. We have tested the model for various markets, and the results are satisfactory.

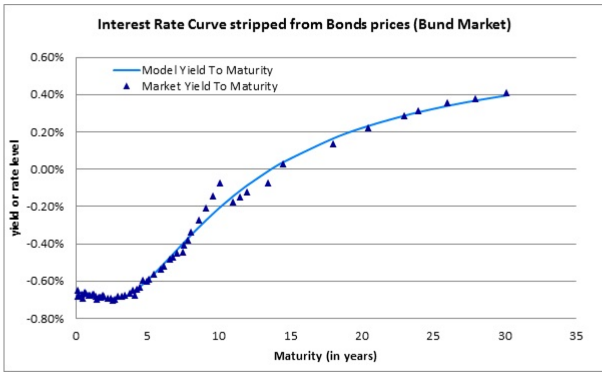
References

[1] Gee P. and Roberts-Sklar Matt (2016). Yield Curves, MFAD induction session. Bank Of England.

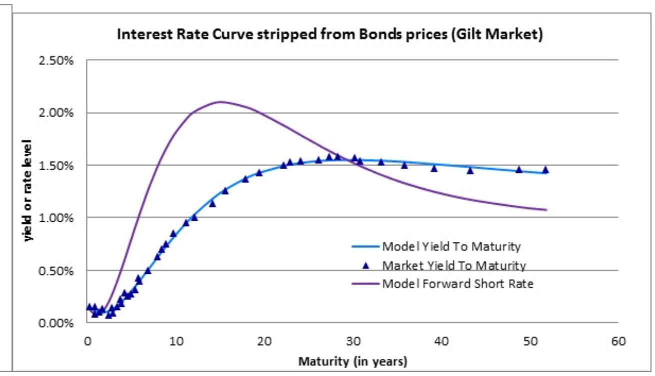
[2] Kouokap Youmbi D. (2012). Pricing of Options on Forward Bonds and Constant Maturity Treasury (CMT): A Monte Carlo Approach. Working paper. SSRN

Appendix: Numerical tests and results

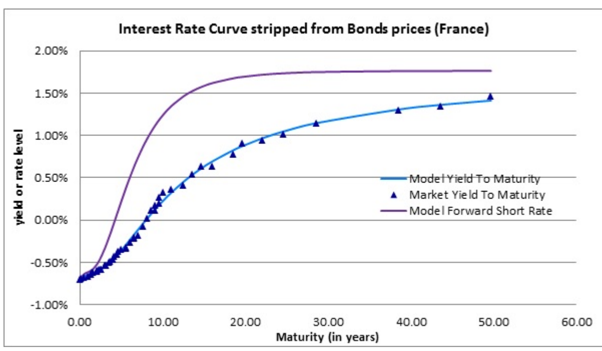
Computation Results



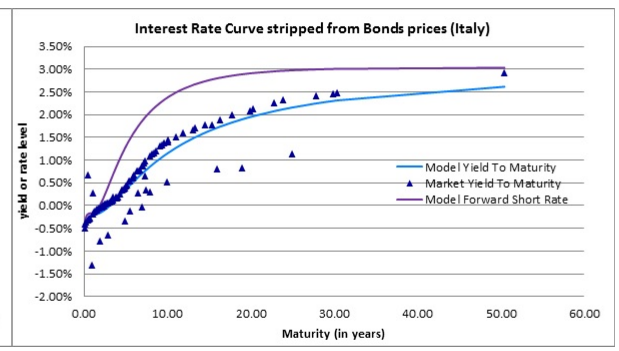
(a) Germany-Bund



(b) UK-Gilt



(c) France



(d) Italy

Figure 4: Calibration Fit Europe

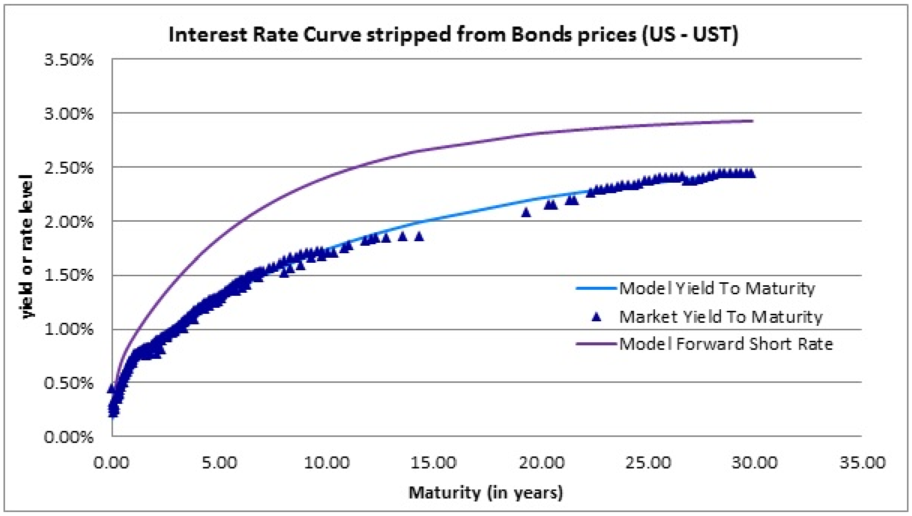


Figure 5: Calibration Fit USD - UST

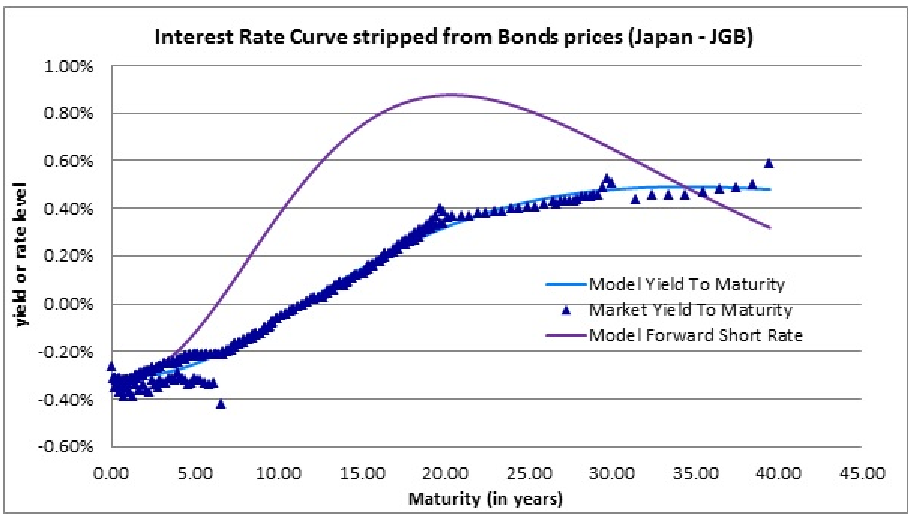
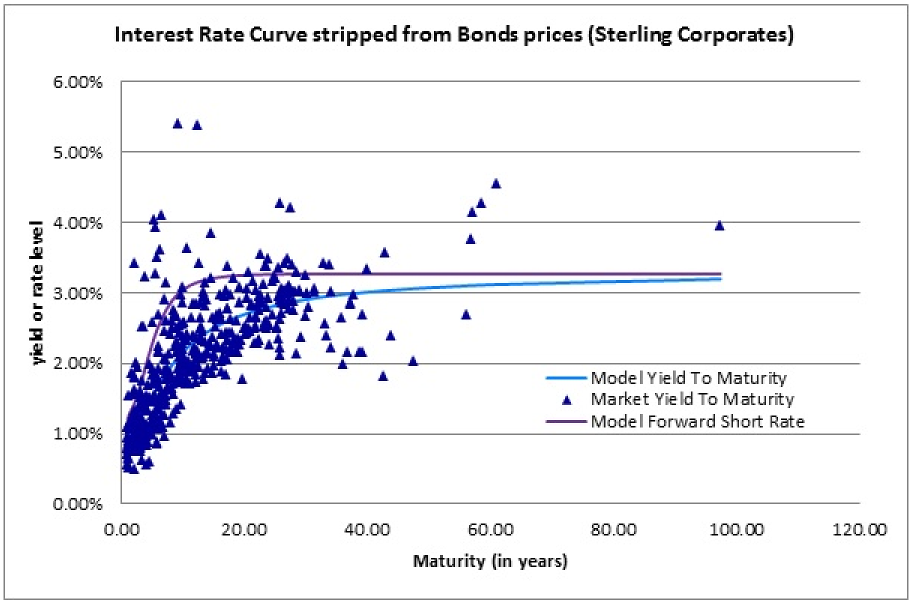
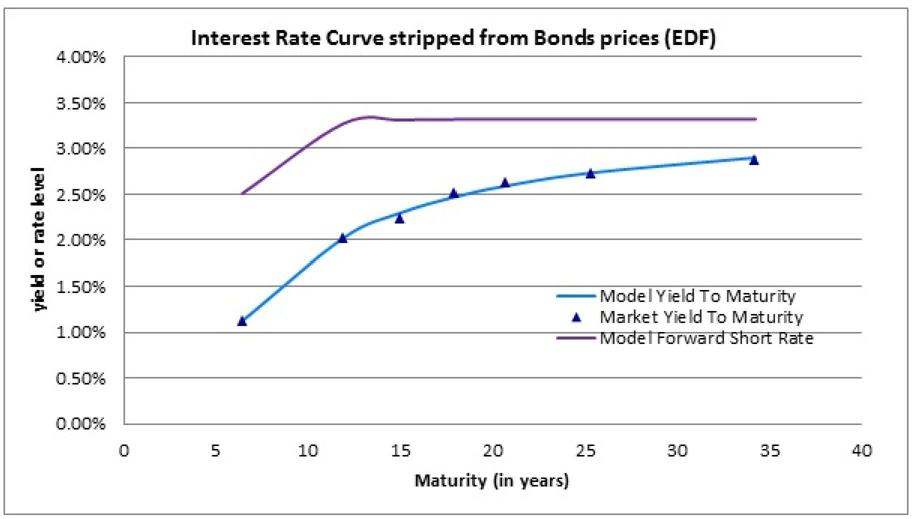


Figure 6: Calibration Fit Japan - JGB



(a) Sterling Corporates



(b) EDF

Figure 7: Calibration Fit: Corporates